GENERATING ULTRAFILTERS IN A REASONABLE WAY

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Abstract. We continue investigations of *reasonable ultrafilters* on uncountable cardinals defined in Shelah [8]. We introduce a general scheme of generating a filter on λ from filters on smaller sets and we investigate the combinatorics of objects obtained this way.

0. Introduction

Reasonable ultrafilters were introduced in Shelah [8] in order to suggest a line of research that would repeat in some sense the beautiful theory created around the notion of *P–points on ω*. If we are interested in generalizing P–points, but we do not want to deal with large cardinals, we have to be somewhat creative in re-interpreting the property that *any countable family of members of the ultrafilter has a pseudo-intersection in the ultrafilter*. An interesting way of doing this is to look at the ways an ultrafilter on an uncountable cardinal λ can be obtained from λ–sequences of objects on smaller cardinals. The general scheme for this approach is motivated by [4, §5,6] and it is presented in Definition 1.2. In this context the P–pointness of an ultrafilter may be re-interpreted as $(<\lambda^+)$–directness of its generating system (see 1.3).

As in [8], when working with ultrafilters on λ, we want to concentrate on those which are *very non-normal*. Thus very often we ask ourselves questions concerning *weak reasonability* of the ultrafilter obtained from a generating system, and the following property is always of interest in this paper.

Definition 0.1 (Shelah [8, Def. 1.4])**.** (1) We say that a uniform ultrafilter D on λ is *weakly reasonable* if for every non-decreasing unbounded function $f\in {}^\lambda\lambda$ there is a club C of λ such that

$$\bigcup\{[\delta,\delta+f(\delta)):\delta\in C\}\notin D.$$

(2) Let D be an ultrafilter on λ, $C\subseteq\lambda$ be a club and let $\langle\delta_\xi:\xi<\lambda\rangle$ be the increasing enumeration of $C\cup\{0\}$. We define

$$D/C=\big\{A\subseteq\lambda:\bigcup_{\xi\in A}[\delta_\xi,\delta_{\xi+1})\in D\big\}.$$

(It is an ultrafilter on λ.) D/C will be called *the quotient of D by C*.

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Observation 0.2 (Shelah [8, Obs. 1.5])**.** *Let D be a uniform ultrafilter on a regular uncountable cardinal λ. Then the following conditions are equivalent:*

(A) *D is weakly reasonable,*
(B) *for every increasing continuous sequence $\langle\delta_\xi : \xi<\lambda\rangle\subseteq\lambda$ there is a club C^* of λ such that*

$$\bigcup\big\{[\delta_\xi,\delta_{\xi+1}) : \xi\in C^*\big\}\notin D,$$

(C) *for every club C of λ the quotient D/C does not extend the filter generated by clubs of λ.*

This paper continues Shelah [8] and Rosłanowski and Shelah [3], but it is essentially self contained. In the first section we present our key definitions introducing *systems of local filters* and partial orders $\mathbb{Q}^*_\lambda(\mathcal{F})$, $\mathbb{Q}^0_\lambda(\mathcal{F})$ associated with them. We explain how those partial orders can be made $(<\lambda^+)$–complete (in 1.6, 1.8) and we show that ultrafilters generated by sufficiently directed generating systems are weakly reasonable, unless they are produced from a measurable ultrafilter (see 1.9). The second section is concerned with the *full system $\mathcal{F}^{\rm ult}$ of local ultrafilters* and the ultrafilters on λ generated by $H\subseteq\mathbb{Q}^*_\lambda(\mathcal{F}^{\rm ult})$. We show that there may be weakly reasonable ultrafilters on λ generated by some $H'\subseteq\mathbb{Q}^0_\lambda(\mathcal{F})$ which cannot be obtained by use of $\mathcal{F}^{\rm ult}$ (see 2.3). Furthermore, we introduce more properties of families $H\subseteq\mathbb{Q}^*_\lambda(\mathcal{F}^{\rm ult})$ which are useful in generating ultrafilters on λ. In the third section we are interested in a system $\mathcal{F}^{\rm pr}$ of local filters and its relation to generating numbers (in standard sense) of filters on λ (see 3.6, 3.8). Finally, in the last section we show that the inaccessibility of λ in the assumptions of [8, Prop. 1.6(1)] is needed: consistently, there is a very reasonable ultrafilter D on ω_1 such that Odd has a winning strategy in $\Game_D$ (see 4.8).

Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [1]).

(1) Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet $(\alpha,\beta,\gamma,\delta\ldots)$ and also by i,j (with possible sub- and superscripts). Cardinal numbers will be called κ,λ,μ (with possible sub- and superscripts). **λ is always assumed to be an uncountable regular cardinal**.
(2) For two sequences η,ν we write $\nu\vartriangleleft\eta$ whenever ν is a proper initial segment of η, and $\nu\trianglelefteq\eta$ when either $\nu\vartriangleleft\eta$ or $\nu=\eta$. The length of a sequence η is denoted by ${\rm lh}(\eta)$.
(3) We will use letters D,E,F and d (with possible indexes) to denote filters on various sets. Typically, D will be a filter on λ (possibly an ultrafilter), while E,F will stand for filters on smaller sets. Also, in most cases d will be an ultrafilter on a set of size less than λ.
For a filter F of subsets of a set A, the family of all F–positive subsets of A is called F^+. (So $B\in F^+$ if and only if $B\subseteq A$ and $B\cap C\neq\emptyset$ for all $C\in F$.)
(4) In forcing we keep the older convention that *a stronger condition is the larger one.* For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$).

1. Generating a filter from systems of local filters

Here we present the general scheme of generating a filter on a **regular uncountable** cardinal λ by using smaller filters. Our approach is slightly different from the one in [8, §2] and/or [3, §1], but the difference is notational only (see 1.3 below).

Definition 1.1. (1) A *system of local filters on* λ is a family $\mathcal{F}$ such that

- all members of $\mathcal{F}$ are triples (α, Z, F) such that $Z \subseteq \lambda$, $|Z| < \lambda$, $\alpha = \min(Z)$ and F is a proper filter on Z,
- the set $\big\{\alpha < \lambda : \big(\exists Z, F\big)\big((\alpha, Z, F) \in \mathcal{F}\big)\big\}$ is unbounded in λ.

If above for every $(\alpha, Z, F) \in \mathcal{F}$, the set Z is infinite and F is a non-principal ultrafilter on Z, then we say that $\mathcal{F}$ *is a system of local non-principal ultrafilters.*

(2) More generally, if Ψ is a property of filters, then a *system of local* Ψ*–filters on* λ is a system of local filters $\mathcal{F}$ such that for every $(\alpha, Z, F) \in \mathcal{F}$, the filter F has the property Ψ. The *full system of local* Ψ*–filters* is the family of *all* triples (α, Z, F) such that $\alpha < \lambda$, $\alpha \in Z \subseteq \lambda \setminus \alpha$, $|Z| < \lambda$ and F is a proper filter on Z with the property Ψ (assuming that it forms a system of local filters). The full system of local non-principal ultrafilters on λ is denoted by $\mathcal{F}^{\rm ult}_\lambda$ or just $\mathcal{F}^{\rm ult}$ (if λ is understood).

The next definition introduces the filters generated by some families of local filters. As we have said in the introduction, our motivations have roots in forcings with norms and this suggested us to use sometimes a forcing-like notation (e,g, $\mathbb{Q}^*_\lambda$) similar to that of [4]. It is also worth noticing that some families of generators may be used as forcing notions - for instance $(\mathbb{Q}^0_\lambda, \leq^*)$ is the forcing used in the end of [3, Sec. 1].

Definition 1.2. Let $\mathcal{F}$ be a system of local filters on λ.

(1) We let $\mathbb{Q}^*_\lambda(\mathcal{F})$ be the family of all sets $r \subseteq \mathcal{F}$ such that

$$\big(\forall \xi < \lambda\big)\big(|\{(\alpha, Z, F) \in r : \alpha = \xi\}| < \lambda\big) \quad \text{and} \quad |r| = \lambda.$$

For $r \in \mathbb{Q}^*_\lambda(\mathcal{F})$ we define

$$\mathrm{fil}(r) = \big\{A \subseteq \lambda : \big(\exists \varepsilon < \lambda\big)\big(\forall (\alpha, Z, F) \in r\big)\big(\varepsilon \leq \alpha \ \Rightarrow\ A \cap Z \in F\big)\big\},$$

and we define a binary relation $\leq^* = \leq^*_{\mathcal{F}}$ on $\mathbb{Q}^*_\lambda(\mathcal{F})$ by
$r_1 \leq^*_{\mathcal{F}} r_2$ if and only if $(r_1, r_2 \in \mathbb{Q}^*_\lambda(\mathcal{F})$ and$)$ $\mathrm{fil}(r_1) \subseteq \mathrm{fil}(r_2)$.

(2) We say that an $r \in \mathbb{Q}^*_\lambda(\mathcal{F})$ is *strongly disjoint* if and only if

- $\big(\forall \xi < \lambda\big)\big(|\{(\alpha, Z, F) \in r : \alpha = \xi\}| < 2\big)$, and
- $\big(\forall (\alpha_1, Z_1, F_1), (\alpha_2, Z_2, F_2) \in r\big)\big(\alpha_1 < \alpha_2 \ \Rightarrow\ Z_1 \subseteq \alpha_2\big)$.

We let $\mathbb{Q}^0_\lambda(\mathcal{F})$ be the collection of all strongly disjoint elements of $\mathbb{Q}^*_\lambda(\mathcal{F})$.

(3) We write $\mathbb{Q}^*_\lambda, \mathbb{Q}^0_\lambda$ for $\mathbb{Q}^*_\lambda(\mathcal{F}^{\rm ult}), \mathbb{Q}^0_\lambda(\mathcal{F}^{\rm ult})$, respectively (where, remember, $\mathcal{F}^{\rm ult}$ is the full system of local non-principal ultrafilters).

(4) For a set $H \subseteq \mathbb{Q}^*_\lambda(\mathcal{F})$ we let $\mathrm{fil}(H) = \bigcup\big\{\mathrm{fil}(r) : r \in H\big\}$.

Remark 1.3. (1) Note that if $r \in \mathbb{Q}^0_\lambda$ then there is $r' \in \mathbb{Q}^0_\lambda$ such that $\mathrm{fil}(r') = \mathrm{fil}(r)$ and for some club C of λ we have

$$\big\{(\alpha, Z) : \big(\exists d\big)\big((\alpha, Z, d) \in r'\big)\big\} = \big\{(\alpha, [\alpha, \beta)) : \alpha \in C \ \&\ \beta = \min\big(C \setminus (\alpha + 1)\big)\big\}.$$

Thus $\mathbb{Q}^0_\lambda$ is essentially the same as the one defined in [8, Def. 2.5].

(2) If $H \subseteq \mathbb{Q}^*_\lambda(\mathcal{F})$ is $\leq^*$–directed, then $D = \mathrm{fil}(H)$ is a filter on λ extending the filter of co-bounded sets. We may say the that *the filter D is generated by H* or that *H is the generating system for D.*

Definition 1.4. Suppose that

(a) X is a non-empty set and F is a filter on X,
(b) F_x is a filter on a set Z_x (for $x \in X$).

We let

$$\bigoplus_{x\in X}^{F} F_x = \big\{A \subseteq \bigcup_{x\in X} Z_x : \{x \in X : Z_x \cap A \in F_x\} \in F\big\}.$$

(Clearly, $\bigoplus\limits_{x\in X}^{F} F_x$ is a filter on $\bigcup\limits_{x\in X} Z_x$.) If X is a linearly ordered set (e.g. it is a set of ordinals) with no maximal element and F is the filter of all co-bounded subsets of X, then we will write $\bigoplus\limits_{x\in X} F_x$ instead of $\bigoplus\limits_{x\in X}^{F} F_x$.

Proposition 1.5 (Cf. [8, Prop. 2.9])**.** (1) *Let $\mathcal{F}$ be a system of local filters on λ and $p, q \in \mathbb{Q}^*_\lambda(\mathcal{F})$. Then $p \leq^* q$ if and only if there is $\varepsilon < \lambda$ such that*

$$\big(\forall (\alpha, Z, F) \in q\big)\big(\forall A \in F^+\big)\big(\alpha > \varepsilon \ \Rightarrow\ \big(\exists (\alpha', Z', F') \in p\big)\big(A \cap Z' \in (F')^+\big)\big).$$

(2) *Let $p, q \in \mathbb{Q}^*_\lambda$. Then the following are equivalent:*
  (a) *$p \leq^* q$,*
  (b) *there is $\varepsilon < \lambda$ such that*

$$\big(\forall (\alpha, Z, d) \in q\big)\big(\forall A \in d\big)\big(\alpha > \varepsilon \ \Rightarrow\ \big(\exists (\alpha', Z', d') \in p\big)\big(A \cap Z' \in d'\big)\big),$$

  (c) *there is $\varepsilon < \lambda$ such that if $(\alpha, Z, d) \in q$, $\varepsilon \leq \alpha$, and $X = \big\{(\xi, Z', d') \in p : Z' \cap Z \neq \emptyset\big\}$, then $X \neq \emptyset$ and there is an ultrafilter e on X such that*

$$d = \big\{A \cap Z : A \in \bigoplus^{e} \{d' : (\exists \xi, Z')((\xi, Z', d') \in X)\}\big\}.$$

The quasi-orders $(\mathbb{Q}^*_\lambda, \leq^*)$ and $(\mathbb{Q}^0_\lambda, \leq^*)$ are $({<}\lambda^+)$–complete (cf. [8, Prop. 2.3(3)]). Moreover, by essentially the same argument we may show the following observation.

Proposition 1.6. *Assume that $\mathcal{F}$ is a system of local filters on λ such that*

$(\oplus)^{\rm sum}_{\mathcal{F}}$ *if $\kappa < \lambda$ is an infinite cardinal and a sequence $\langle (\alpha_\xi, Z_\xi, F_\xi) : \xi < \kappa\rangle \subseteq \mathcal{F}$ satisfies*

$$(\forall \xi < \zeta < \kappa)(Z_\xi \subseteq \alpha_\zeta),$$

then for some uniform filter F on κ we have $\big(\alpha_0, \bigcup\limits_{\xi<\kappa} Z_\xi, \bigoplus\limits_{\xi<\kappa}^{F} F_\xi\big) \in \mathcal{F}$.

Then both $\mathbb{Q}^*_\lambda(\mathcal{F})$ and $\mathbb{Q}^0_\lambda(\mathcal{F})$ are $({<}\lambda^+)$–complete (with respect to $\leq^*$).

It is worth noticing that in general $\mathbb{Q}^*_\lambda(\mathcal{F})$ and/or $\mathbb{Q}^0_\lambda(\mathcal{F})$ do not have to be even σ–complete. For instance, consider the full system of co-bounded filters $\mathcal{F}_0$; it consists of all triples (α, Z, F) such that $\alpha \in Z \subseteq \lambda \setminus \alpha$, $|Z| < \lambda$, $\sup(Z) \notin Z$ and F is the filter of all co-bounded subsets of Z. Let C consist of all ordinals $\alpha < \lambda$

divisible by $\omega\cdot\omega$, and for $\alpha\in C$ and $m<\omega$ let $Z^\alpha_m=[\alpha+m\cdot\omega,\alpha+m\cdot\omega+\omega)$ and F^α_m be the filter of co-bounded subsets of Z^α_m. For $n<\omega$ put

$$p_n=\big\{(\alpha+m\cdot\omega,Z^\alpha_m,F^\alpha_m):\alpha\in C\ \&\ 0<m<\omega\ \&\ 2^n|m\big\}.$$

Clearly $p_n\in\mathbb{Q}^0_\lambda(\mathcal{F}_0)$ and $p_n\leq^* p_{n+1}$ for all $n<\omega$. One may easily verify that the sequence $\langle p_n:n<\omega\rangle$ has no $\leq^*$–upper bound in $\mathbb{Q}^*_\lambda(\mathcal{F}_0)$.

There is a natural procedure which for a given system $\mathcal{F}$ of local filters on λ generates a system $\mathcal{F}^*\supseteq\mathcal{F}$ satisfying the condition $(\oplus)^{\rm sum}_{\mathcal{F}}$ of 1.6 (so then $\mathbb{Q}^*_\lambda(\mathcal{F}^*)$ and $\mathbb{Q}^0_\lambda(\mathcal{F}^*)$ are suitably complete).

Definition 1.7. Assume that

(a) $\mathcal{F}$ is a system of local filters on λ,
(b) $\bar{E}=\langle E_\kappa:\kappa$ is a cardinal $\&\ \aleph_0\leq\kappa<\lambda\rangle$, where each E_κ is a uniform filter on κ.

We define:

(1) An $(\bar{E},\mathcal{F})$*–block* is a pair $(T,\bar{D})$ such that
- $T\subseteq{}^{<\omega}\lambda$ is a well–founded tree,
- if $\eta\in T\setminus\max(T)$, then $\{\xi<\lambda:\eta^\frown\langle\xi\rangle\in T\}=\kappa$ for some infinite cardinal $\kappa<\lambda$,
- $\bar{D}=\langle(\alpha_\eta,Z_\eta,F_\eta):\eta\in\max(T)\rangle\subseteq\mathcal{F}$,
- if $\eta,\nu\in\max(T)$ and $\eta<_{\rm lex}\nu$, then $Z_\eta\subseteq\alpha_\nu$ (where $<_{\rm lex}$ is the lexicographic order of T).

(2) By induction on the rank of the tree T, for an $(\bar{E},\mathcal{F})$–block $(T,\bar{D})$ we define a filter $\bar{D}(T)$ on $\bigcup\{Z_\eta:\eta\in\max(T)\}$ (where $\bar{D}=\langle(\alpha_\eta,Z_\eta,F_\eta):\eta\in\max(T)\rangle$).
- If ${\rm rk}(T)=0$, i.e., $T=\{\langle\rangle\}$ then $\bar{D}(T)=F_{\langle\rangle}$.
- Suppose ${\rm rk}(T)>0$. Let $\kappa=\{\xi<\lambda:\langle\xi\rangle\in T\}$ (so $\aleph_0\leq\kappa<\lambda$ is a cardinal). For $\xi<\kappa$ we put

$$T^\xi=\{\nu\in{}^{<\omega}\lambda:\langle\xi\rangle^\frown\nu\in T\}\ \text{ and }\ \bar{D}^\xi=\langle(\alpha_\eta,Z_\eta,F_\eta):\eta\in\max(T)\ \&\ \eta(0)=\xi\rangle.$$

Plainly, each $(T^\xi,\bar{D}^\xi)$ is an $(\bar{E},\mathcal{F})$–block (and ${\rm rk}(T^\xi)<{\rm rk}(T)$). We define

$$\bar{D}(T)=\bigoplus_{\xi<\kappa}^{E_\kappa}\bar{D}^\xi(T^\xi).$$

(3) *The $\bar{E}$–closure of* $\mathcal{F}$ is the family of all triples (α,Z,D) such that $\alpha<\lambda$ and for some $(\bar{E},\mathcal{F})$–block $(T,\bar{D})$ we have

$$Z=\bigcup\{Z_\eta:\eta\in\max(T)\}\quad\text{and}\quad D=\bar{D}(T)\quad\text{and}\quad\alpha=\min(Z)$$

(where $\bar{D}=\langle(\alpha_\eta,Z_\eta,F_\eta):\eta\in\max(T)\rangle$).

Proposition 1.8. *Assume that*

(a) *$\mathcal{F}$ is a system of local filters on λ,*
(b) *$\bar{E}=\langle E_\kappa:\aleph_0\leq\kappa<\lambda\ \&\ \kappa$ is a cardinal $\rangle$, where each E_κ is a uniform filter on κ.*

Then the $\bar{E}$–closure of $\mathcal{F}$ is a system of local filters extending $\mathcal{F}$ and satisfying the condition $(\oplus)^{\rm sum}_{\mathcal{F}}$ of 1.6.

Suppose that a system $\mathcal{F}'$ of local filters on λ includes all triples $(\alpha,\{\alpha\},d)$, where $\alpha<\lambda$ and d is the principal ultrafilter on $\{\alpha\}$. For a set $A\subseteq\lambda$ let $p_A=\{(\alpha,\{\alpha\},d)\in\mathcal{F}':\alpha\in A\}\in\mathbb{Q}^0_\lambda(\mathcal{F}')$. Note that $p_A\cap p_B=p_{A\cap B}$, so easily if D is a filter on λ extending the co-bounded filter, then $H^D\stackrel{\rm def}{=}\{p_A:A\in D\}$ is a $\leq^*$–directed family and $\mathrm{fil}(H^D)=D$. If D is a normal filter on λ, then H^D will be also $({<}\lambda^+)$–directed (with respect to $\leq^*$). Consequently, if λ is a measurable cardinal, then we may find a system $\mathcal{F}$ of local filters on λ and a $({<}\lambda^+)$–directed family $H\subseteq\mathbb{Q}^0_\lambda(\mathcal{F})$ such that $\mathrm{fil}(H)$ is an ultrafilter including all club subsets of λ (so $\mathrm{fil}(H)$ is not weakly reasonable). However, to have a quite directed family H such that $\mathrm{fil}(H)$ is a non-reasonable ultrafilter we do need a measurable cardinal.

Theorem 1.9. *Suppose that $\mathcal{F}$ is a system of local filters on λ, $\kappa\leq\lambda$ and $H\subseteq\mathbb{Q}^*_\lambda(\mathcal{F})$ is a $({<}\kappa)$–directed family such that $\mathrm{fil}(H)$ is an ultrafilter. If $\mathrm{fil}(H)$ is not weakly reasonable, then for some club C^* of λ the quotient ultrafilter $\mathrm{fil}(H)/C^*$ is $({<}\kappa)$–complete and it contains all clubs of λ.*

Proof. Assume that the family $H\subseteq\mathbb{Q}^*_\lambda(\mathcal{F})$ is $({<}\kappa)$–directed and $\mathrm{fil}(H)$ is an ultrafilter which is not weakly reasonable. Let $\bar{\delta}=\langle\delta_\xi:\xi<\lambda\rangle$ be an increasing continuous sequence of ordinals below λ such that $\delta_0=0$ and for every club $C\subseteq\lambda$ we have that $\bigcup\big\{[\delta_\xi,\delta_{\xi+1}):\xi\in C\big\}\in\mathrm{fil}(H)$. Now, for a club C of λ and $p\in H$ put

$$S(p,C)=\big\{\xi\in C:\big(\exists(\alpha,Z,F)\in p\big)\big([\delta_\xi,\delta_{\xi+1})\cap Z\in F^+\big)\big\}.$$

Claim 1.9.1. *For every club $C\subseteq\lambda$ and $p\in H$, the set $S(p,C)$ is stationary.*

Proof of the Claim. Assume towards contradiction that $S(p,C)$ is non-stationary. So we may choose a club $C'\subseteq C$ of λ such that

$(*)_1$ $\big(\forall\xi\in C'\big)\big(\forall(\alpha,Z,F)\in p\big)\big(Z\setminus[\delta_\xi,\delta_{\xi+1})\in F\big)$.

Pick a club $C''\subseteq C'$ such that

$(*)_2$ $\big(\forall(\alpha,Z,F)\in p\big)\big(\forall\xi\in C''\big)\big(\alpha<\delta_\xi\ \Rightarrow\ Z\subseteq\delta_\xi\big)$.

By the choice of $\bar{\delta}$ we know that $\bigcup\{[\delta_\xi,\delta_{\xi+1}):\xi\in C''\}\in\mathrm{fil}(H)$, so necessarily $\bigcup\{[\delta_\xi,\delta_{\xi+1}):\xi\in C''\}\in\big(\mathrm{fil}(p)\big)^+$. Thus we may pick $(\alpha,Z,F)\in p$ and $\xi\in C''$ such that $Z\cap[\delta_\xi,\delta_{\xi+1})\in F^+$ (remember $(*)_2$), contradicting $(*)_1$. □

Claim 1.9.2. (1) *If $p\leq^* q$, $p,q\in H$ and $C'\subseteq C$ are clubs of λ, then $|S(q,C')\setminus S(p,C)|<\lambda$.*

(2) *If $A\subseteq\lambda$, then there are $p\in H$ and a club $C\subseteq\lambda$ such that either $S(p,C)\subseteq A$ or $S(p,C)\subseteq\lambda\setminus A$.*

Proof of the Claim. (1) Pick $\gamma<\lambda$ so that

$$\big(\forall(\alpha,Z,F)\in q\big)\big(\forall A\in F^+\big)\big(\alpha>\gamma\ \Rightarrow\ (\exists(\alpha',Z',F')\in p)(A\cap Z'\in(F')^+)\big)$$

(remember 1.5) and let $\gamma^*<\lambda$ be such that $\gamma<\gamma^*$ and $\big(\forall(\alpha,Z,F)\in q\big)\big(\alpha\leq\gamma\ \Rightarrow\ Z\subseteq\gamma^*\big)$. Suppose that $\xi\in S(q,C')\setminus\gamma^*$. Then $\xi\in C'\subseteq C$ and there is $(\alpha,Z,F)\in q$ such that $[\delta_\xi,\delta_{\xi+1})\cap Z\in F^+$. Since $\delta_\xi\geq\xi\geq\gamma^*$, we also have $\alpha>\gamma$ and hence there is $(\alpha',Z',F')\in p$ such that $[\delta_\xi,\delta_{\xi+1})\cap Z\cap Z'\in(F')^+$. Hence we may conclude that $\xi\in S(p,C)$.

(2) Assume $A\subseteq\lambda$. Let $A^*=\bigcup\big\{[\delta_\xi,\delta_{\xi+1}):\xi\in A\big\}$. Since $\mathrm{fil}(H)$ is an ultrafilter, then either A^* or $\lambda\setminus A^*$ belongs to it. Suppose $A^*\in\mathrm{fil}(p)$ for some $p\in H$. Pick a club $C\subseteq\lambda$ such that

($\odot$) if $(\alpha, Z, F) \in p$ and $(\sup(Z)+1) \cap C \neq \emptyset$, then $A^* \cap Z \in F$.

Suppose $\xi \in S(p,C)$, so $\xi \in C$ and for some $(\alpha, Z, F) \in p$ we have $[\delta_\xi, \delta_{\xi+1}) \cap Z \in F^+$. It follows from ($\odot$) that $A^* \cap Z \in F$ and therefore $\xi \in A$. Thus $S(p,C) \subseteq A$.

If $\lambda \setminus A^* \in \mathrm{fil}(H)$, then we proceed in an analogous manner. $\square$

Let

$$D = \big\{ A \subseteq \lambda : |S(p,C) \setminus A| < \lambda \text{ for some } p \in H \text{ and a club } C \subseteq \lambda \ \big\}.$$

It follows from 1.9.1 that all members of D are stationary and since H is directed we may use 1.9.2(1) to argue that D is a filter on λ. By 1.9.2(2) we see that D is an ultrafilter on λ (so it also contains all clubs as its members are stationary). Since H is $({<}\kappa)$–directed and the intersection of ${<}\kappa$ many clubs is a club, we may conclude from 1.9.2(1) that D is a $({<}\kappa)$–complete ultrafilter.

Let $C^* = \{\delta_\xi : \xi < \lambda\}$ (so it is a club of λ). To complete the proof of the theorem we are going to show that $D = \mathrm{fil}(H)/C^*$. Since we already know that D is an ultrafilter, it is enough to show that $S(p,C) \in \mathrm{fil}(H)/C^*$ for every $p \in H$ and a club $C \subseteq \lambda$. So let $C \subseteq \lambda$ be a club, $p \in H$ and $S^* = \bigcup \big\{ [\delta_\xi, \delta_{\xi+1}) : \xi \in S(p,C) \big\}$. If $S^* \in \mathrm{fil}(H)$, then we are done, so assume that $S^* \notin \mathrm{fil}(H)$. Since $\mathrm{fil}(H)$ is an ultrafilter and H is directed, we may find $q \in H$ such that $p \leq^* q$ and $\lambda \setminus S^* \in \mathrm{fil}(q)$. Let $\gamma < \lambda$ be such that

$$\big(\forall (\alpha, Z, F) \in q\big)\big(\gamma \leq \sup(Z) \ \Rightarrow \ Z \setminus S^* \in F\big).$$

Since $|S(q,C) \setminus S(p,C)| < \lambda$, we may pick $\xi \in S(q,C) \cap S(p,C)$ such that $\xi > \gamma$. Then $[\delta_\xi, \delta_{\xi+1}) \subseteq S^*$ but also there is $(\alpha, Z, F) \in q$ such that $[\delta_\xi, \delta_{\xi+1}) \cap Z \in F^+$, and thus also $S^* \cap Z \in F^+$. However, $\sup(Z) \geq \delta_\xi > \gamma$, so $Z \setminus S^* \in F$ by the choice of γ, a contradiction showing that $S^* \in \mathrm{fil}(H)$ as required. $\square$

2. Systems of local ultrafilters

In this section we are interested in the full system $\mathcal{F}^{\mathrm{ult}}$ of local ultrafilters on λ and $\mathbb{Q}^*_\lambda, \mathbb{Q}^0_\lambda$. The first question that one may ask is whether weakly reasonable ultrafilters on λ generated by some $H \subseteq \mathbb{Q}^0_\lambda(\mathcal{F})$ can be obtained by the use of $\mathbb{Q}^0_\lambda$. It occurs that it does matter which system of local filters we are using.

Definition 2.1. A filter F on a set Z is called *an unultra filter,* if for every $A \in F^+$ there is $B \subseteq A$ such that both $B \in F^+$ and $A \setminus B \in F^+$. The full system of local unultra filters on λ will be denoted by $\mathcal{F}^{\mathrm{unu}}$. (Thus $\mathcal{F}^{\mathrm{unu}}$ consists of all triples (α, Z, F) such that $\emptyset \neq Z \subseteq \lambda$, $|Z| < \lambda$, $\alpha = \min(Z)$ and F is an unultra filter on Z.)

Observation 2.2. (1) *If F is an unultra filter on Z, $A \in F^+$, then $F + A \stackrel{\mathrm{def}}{=} \{B \subseteq Z : B \cup (Z \setminus A) \in F\}$ is an unultra filter.*

(2) *Suppose that ξ is a limit ordinal, $\{Z_\zeta : \zeta < \xi\}$ is a family of pairwise disjoint non-empty sets, F_ζ is a filter on Z_ζ (for $\zeta < \xi$). Then $\bigoplus\limits_{\zeta<\xi} F_\zeta$ is an unultra filter on $\bigcup\limits_{\zeta<\xi} Z_\xi$. (Remember the convention declared in the last sentence of Definition 1.4.)*

Theorem 2.3. *Assume $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$. There exists a $\leq^*$–increasing sequence $\langle p_\xi : \xi < \lambda^+ \rangle \subseteq \mathbb{Q}^0_\lambda(\mathcal{F}^{\mathrm{unu}})$ such that*

(a) $\mathrm{fil}\big(\{p_\xi : \xi < \lambda^+\}\big)$ *is a weakly reasonable ultrafilter on λ, but*

(b) *there is no* $p\in\mathbb{Q}^0_\lambda$ *with* $\mathrm{fil}(p)\subseteq\mathrm{fil}\big(\{p_\xi:\xi<\lambda^+\}\big)$.

Proof. Fix enumerations

- $\langle Y_\zeta:\zeta<\lambda^+\ \&\ \zeta$ is limit $\rangle$ of all subsets of λ, and
- $\langle r_\zeta:\zeta<\lambda^+\ \&\ \zeta$ is limit $\rangle$ of $\mathbb{Q}^0_\lambda$, and
- $\langle\bar\delta^\zeta:\zeta<\lambda^+\ \&\ \zeta$ is limit $\rangle$ of all increasing continuous sequences of ordinals below λ, $\bar\delta^\zeta=\langle\delta^\zeta_\alpha:\alpha<\lambda\rangle$.

By induction on $\xi<\lambda^+$ we choose $p_\xi\in\mathbb{Q}^0_\lambda(\mathcal{F}^{\rm unu})$ so that the following conditions are satisfied for every limit ordinal $\zeta<\lambda^+$.

(o) For $n<\omega$, the element $p_n\in\mathbb{Q}^0_\lambda(\mathcal{F}^{\rm unu})$ is

$\big\{(\alpha,Z_\alpha,F_\alpha):\alpha<\lambda$ is limit, $Z_\alpha=[\alpha,\alpha{+}\omega)$ and F_α is the filter of co-finite subsets of $Z_\alpha\big\}$.

(i) If $\mathrm{cf}(\zeta)<\lambda$, then for some increasing and cofinal in ζ sequence $\langle\zeta_i:i<\mathrm{cf}(\zeta)\rangle$, for every $(\alpha,Z,F)\in p_\zeta$, there is a sequence $\langle(\alpha_i,Z_i,F_i):i<\mathrm{cf}(\zeta)\rangle$ such that
- $(\alpha_i,Z_i,F_i)\in p_{\zeta_i}$,
- $Z_i\subseteq\alpha_j$ for $i<j<\mathrm{cf}(\zeta)$,
- $Z=\bigcup\limits_{i<\mathrm{cf}(\zeta)}Z_i$ and $F=\bigoplus\{F_i:i<\mathrm{cf}(\zeta)\}$.

(ii) If $\mathrm{cf}(\zeta)=\lambda$, then for some increasing and cofinal in ζ sequence $\langle\zeta_i:i<\lambda\rangle$, if $(\alpha,Z,F)\in p_\zeta$ and $\mathrm{otp}\big(\{\alpha'<\alpha:\big(\exists Z',F'\big)\big((\alpha',Z',F')\in p_\zeta\big)\}\big)=j$, then $(\alpha,Z,F)\in p_{\zeta_j}$ and
$$\big(\forall i<j\big)\big(\forall A\in F^+\big)\big(\exists(\beta,W,D)\in p_{\zeta_i}\big)\big(A\cap W\in D^+\big).$$

(iii) If $|\{(\alpha,Z,F)\in p_\zeta:Y_\zeta\cap Z\in F^+\}|=\lambda$, then
$$p_{\zeta+1}=\big\{\big(\alpha,Z,F+[Y_\zeta\cap Z]\big):(\alpha,Z,F)\in p_\zeta\ \&\ Y_\zeta\cap Z\in F^+\big\},$$
and otherwise $p_{\zeta+1}=\big\{(\alpha,Z,F)\in p_\zeta:Z\setminus Y_\zeta\in F\big\}$.

(iv) $p_{\zeta+2}\subseteq p_{\zeta+1}$ and for some club C of λ, for every $\beta\in C$ we have
- $Z\subseteq\delta^\zeta_\beta$ whenever $(\alpha,Z,F)\in p_{\zeta+2}$, $\alpha<\delta^\zeta_\beta$, and
- $\delta^\zeta_{\beta+1}<\min\big(\alpha\geq\delta^\zeta_\beta:\big(\exists Z,F\big)\big((\alpha,Z,F)\in p_{\zeta+2}\big)\big)$.

(v) $p_{\zeta+3}=\big\{(\alpha,Z,F+A_\alpha):(\alpha,Z,F)\in p_{\zeta+2}\big\}$ where for every $(\alpha,Z,F)\in p_{\zeta+2}$ the set $A_\alpha\in F^+$ is such that $\big(\forall(\beta,Y,d)\in r_\zeta\big)\big(A_\alpha\cap Y\notin d\big)$.

(vi) $p_{\zeta+3+n}=p_{\zeta+3}$ for all $n<\omega$.

(vii) For every $r\in\mathbb{Q}^0_\lambda$ and $\xi<\lambda^+$, if $(\alpha,Z,F)\in p_\xi$ and $A\in F^+$, then there is $A'\subseteq A$ such that
$$A'\in F^+\quad\text{ and }\quad\big(\forall(\beta,Y,d)\in r\big)\big(A'\cap Y\notin d\big).$$

Conditions (o)–(vi) fully describe how the construction is carried out and 1.5+2.2 imply that $\langle p_\xi:\xi<\lambda^+\rangle\subseteq\mathbb{Q}^0_\lambda(\mathcal{F}^{\rm unu})$ is $\leq^*$–increasing. However, we have to argue that the demand in (vii) is satisfied, as it is crucial for the possibility of satisfying the demand in (v). Let $r\in\mathbb{Q}^0_\lambda$. By induction on $\xi<\lambda$ we show that for every $(\alpha,Z,F)\in p_\xi$ we have

$(\boxdot)_{(\alpha,Z,F)}$ if $A\in F^+$, then there is $A'\subseteq A$ such that $A'\in F^+$ and $\big(\forall(\beta,Y,d)\in r\big)\big(A'\cap Y\notin d\big)$.

(For a set A' as above we will say that *it works for* F *and* r.)

Step $\xi<\omega$.

Note that for each limit ordinal $\alpha<\lambda$ there is at most one $(\beta,Y,d)\in r$ such that

$Y \cap [\alpha, \alpha+\omega)$ is infinite. Assume $A \subseteq [\alpha, \alpha+\omega)$ is infinite. Considering any two disjoint infinite sets $A', A'' \subseteq A$ we easily see that one of them must work for the filter of co-finite subsets of $[\alpha, \alpha+\omega)$ and r.

Step $\xi = \zeta + n + 1$, $\zeta < \lambda^+$ is limit, $n < \omega$.
If $(\alpha, Z, F) \in p_{\zeta+n}$, $A \in F^+$, $A \subseteq A^*$ and $A' \subseteq A$ works for F and r, then also A' works for $F + A^*$ and r.

Step $\xi = \zeta < \lambda^+$ is limit.
Suppose that $(\alpha, Z, F) \in p_\zeta$. If $\mathrm{cf}(\zeta) = \lambda$, then $(\alpha, Z, F) \in p_{\xi'}$ for some $\xi' < \zeta$ (see (ii)) so the inductive hypothesis applies directly. So assume that $\mathrm{cf}(\zeta) < \lambda$. Then $Z = \bigcup\limits_{i<\mathrm{cf}(\zeta)} Z_i$ and $F = \bigoplus\{F_i : i < \mathrm{cf}(\zeta)\}$ for some sequence $\langle (\alpha_i, Z_i, F_i) : i < \mathrm{cf}(\zeta)\rangle$ such that

- $(\boxdot)_{(\alpha_i, Z_i, F_i)}$ holds for each $i < \mathrm{cf}(\zeta)$, and
- $Z_i \subseteq \alpha_j$ for $i < j < \mathrm{cf}(\zeta)$.

Let $\alpha^* = \sup(Z)$ and let $A \in F^+$. Now we consider three cases.

Case A: For some $\alpha' < \alpha^*$ we have $\big(\forall (\beta, Y, d) \in r\big)\big(Y \cap [\alpha', \alpha^*) = \emptyset\big)$.
Plainly, the set $A' = A \setminus \alpha'$ works for F and r.

Case B: For some $(\beta, Y, d) \in r$ we have $\beta < \alpha^* \leq \sup(Y)$.
For each $i < \mathrm{cf}(\zeta)$ such that $A \cap Z_i \in (F_i)^+$ choose disjoint sets $A_i^0, A_i^1 \in (F_i)^+$ included in $A \cap Z_i$ (remember each F_i is an unultra filter) and let

$$A^\ell = \bigcup\{A_i^\ell : i < \mathrm{cf}(\zeta) \ \&\ A \cap Z_i \in (F_i)^+\} \setminus \beta \subseteq A$$

(for $\ell < 2$). Both $A^0 \in F^+$ and $A^1 \in F^+$, and one of these two sets works for F and r.

Case C: For each $\alpha' < \alpha^*$ there is $(\beta, Y, d) \in r$ such that $\alpha' < \beta < \sup(Y) < \alpha^*$.
Let $A \in F^+$. Then the set $I = \{i < \mathrm{cf}(\zeta) : A \cap Z_i \in (F_i)^+\}$ is unbounded in $\mathrm{cf}(\zeta)$ and using the assumptions of the current case we may choose an increasing sequence $\langle i_j : j < \mathrm{cf}(\zeta)\rangle \subseteq I$ such that for every $(\beta, Y, d) \in r$ there is at most one $j < \mathrm{cf}(\zeta)$ such that $Z_{i_j} \cap Y \neq \emptyset$. For each $j < \mathrm{cf}(\zeta)$ pick $A'_{i_j} \in (F_{i_j})^+$ included in $A \cap Z_{i_j}$ which works for F_{i_j} and r, and then put $A' = \bigcup\limits_{j<\mathrm{cf}(\zeta)} A'_{i_j}$. $\square$

Problem 2.4. Is it provable in ZFC that for some system $\mathcal{F}$ of local filters on λ there exists a $\leq^*$–directed family $H \subseteq \mathbb{Q}^*_\lambda(\mathcal{F})$ such that

(a) $\mathrm{fil}(H)$ is a weakly reasonable ultrafilter on λ, but
(b) there is no $\leq^*$–directed family $H' \subseteq \mathbb{Q}^0_\lambda(\mathcal{F}^{\mathrm{ult}})$ such that $\mathrm{fil}(H) = \mathrm{fil}(H')$?

The assumption that a generating system $H \subseteq \mathbb{Q}^*_\lambda(\mathcal{F})$ is directed is an easy way to ensure that $\mathrm{fil}(H)$ is a filter on λ. However, if we work with $H \subseteq \mathbb{Q}^*_\lambda$ we may consider alternative ways of guaranteeing this.

Definition 2.5. For $p \in \mathbb{Q}^*_\lambda$ let

$$\Sigma(p) = \big\{(\alpha, Z, d) \in \mathcal{F}^{\mathrm{ult}} : \big(\forall A \in d\big)\big(\exists (\alpha', Z', d') \in p\big)\big(A \cap Z' \in d'\big)\big\}.$$

Observation 2.6. (1) *If $p, q \in \mathbb{Q}^*_\lambda$, then $p \leq^* q$ if and only if $|q \setminus \Sigma(p)| < \lambda$.*

(2) *If $p \in \mathbb{Q}^*_\lambda$ and $(\alpha, Z, d) \in \Sigma(p)$, then for some $\{(\alpha_x, Z_x, d_x) : x \in X\} \subseteq p$ and an ultrafilter e on X we have*

$$d = \big\{A \subseteq Z : A \cap \bigcup_{x\in X} Z_x \in \bigoplus^{e}\{d_x : x \in X\}\big\}.$$

Definition 2.7. We say that a non-empty family $H \subseteq \mathbb{Q}^*_\lambda$ is

(a) *big* if for each $\mathcal{D} \subseteq \mathcal{F}^{\rm ult}$ there is $q \in H$ such that either $q \subseteq \mathcal{D}$ or $q \cap \mathcal{D} = \emptyset$;

(b) *linked* if for each $p_0, \ldots, p_n \in H$, $n < \omega$, we have

$$\big\{\alpha : \big(\exists Z, d\big)\big((\alpha, Z, d) \in \Sigma(p_0) \cap \ldots \cap \Sigma(p_n)\big)\big\}| = \lambda.$$

The property introduced in Definition 2.7(a) resembles the bigness of creating pairs (see [4, Sec. 2.2]), so the use of the term *big* seemed natural. The name *linked* is motivated by Observation 2.8(1) below.

Observation 2.8. (1) *If $H \subseteq \mathbb{Q}^*_\lambda$ is linked, then*

(a) *for each $p_0, \ldots, p_n \in H$, $n < \omega$, there is $q \in \mathbb{Q}^*_\lambda$ which is $\leq^*$–above all $p_0, \ldots, p_n$,*

(b) $\mathrm{fil}(H)$ *has finite intersection property.*

(2) *If $H \subseteq \mathbb{Q}^*_\lambda$ is linked and big, then* $\mathrm{fil}(H)$ *is an ultrafilter on λ.*

For basic information on the ideal $\mathbf{M}^\lambda_{\lambda,\lambda}$ of meager subsets of ${}^\lambda\lambda$ and its covering number we refer the reader e.g. to Matet, Rosłanowski and Shelah [2, §4]. Let us recall the following definition.

Definition 2.9. (1) The space ${}^\lambda\lambda$ is endowed with the topology obtained by taking as basic open sets $\emptyset$ and O_s for $s \in {}^{<\lambda}\lambda$, where $O_s = \{f \in {}^\lambda\lambda : s \subseteq f\}$.

(2) The $(<\lambda^+)$–complete ideal of subsets of ${}^\lambda\lambda$ generated by nowhere dense subsets of ${}^\lambda\lambda$ is denoted by $\mathbf{M}^\lambda_{\lambda,\lambda}$.

(3) $\mathrm{cov}(\mathbf{M}^\lambda_{\lambda,\lambda})$ is the minimal size of a family $\mathcal{A} \subseteq \mathbf{M}^\lambda_{\lambda,\lambda}$ such that $\bigcup \mathcal{A} = {}^\lambda\lambda$.

Theorem 2.10. *Assume that $\lambda = \lambda^{<\lambda} \geq \aleph_1$ and $\mathrm{cov}(\mathbf{M}^\lambda_{\lambda,\lambda}) = 2^\lambda$. Then there exists a linked and big family $H \subseteq \mathbb{Q}^0_\lambda$ such that $\mathrm{fil}(H)$ is a weakly reasonable ultrafilter.*

Proof. The proof is very similar to that of [8, Thm 2.14]. Let χ be a sufficiently large regular cardinal and let $N \prec \mathcal{H}(\chi)$ be such that $|N| = \lambda$ and ${}^{<\lambda}N \subseteq N$. Put $\mathcal{F}^{\rm ult}_N = \mathcal{F}^{\rm ult} \cap N$. We will inductively construct a linked and big family H included in $\mathbb{Q}^0_\lambda(\mathcal{F}^{\rm ult}_N) \subseteq \mathbb{Q}^0_\lambda(\mathcal{F}^{\rm ult})$. The following two claims are the key points of the inductive process. Below, "linked" means "linked as a subfamily of $\mathbb{Q}^*_\lambda$" (i.e., it is the notion introduced in Definition 2.7(b)).

Claim 2.10.1. *Assume that $H_0 \subseteq \mathbb{Q}^*_\lambda(\mathcal{F}^{\rm ult}_N)$ is linked, $|H_0| < \mathrm{cov}(\mathbf{M}^\lambda_{\lambda,\lambda})$, and $\mathcal{D} \subseteq \mathcal{F}^{\rm ult}$. Then there is $q \in \mathbb{Q}^0_\lambda(\mathcal{F}^{\rm ult}_N) \subseteq \mathbb{Q}^0_\lambda$ such that $H_0 \cup \{q\}$ is linked and either $q \subseteq \mathcal{D}$ or $q \cap \mathcal{D} = \emptyset$.*

Proof of the Claim. We consider two cases.
Case A: For every $n < \omega$, $p_0, \ldots, p_n \in H_0$ and $\beta < \lambda$ there is $(\alpha, Z, d) \in \Sigma(p_0) \cap \ldots \cap \Sigma(p_n) \cap \mathcal{D} \cap \mathcal{F}^{\rm ult}_N$ such that $\beta < \alpha$.
Let $\mathcal{T}_0$ be the family of all sequences η such that

(i) $\mathrm{lh}(\eta) < \lambda$,

(ii) if $\xi < \mathrm{lh}(\eta)$, then $\eta(\xi) \in \mathcal{D} \cap \mathcal{F}^{\rm ult}_N$,
(iii) if $\xi < \xi' < \mathrm{lh}(\eta)$, $\eta(\xi) = (\alpha, Z, d)$, $\eta(\xi') = (\alpha', Z', d')$, then $Z \subseteq \alpha'$.

It follows from the assumptions of the current case that $\mathcal{T}_0$ is a λ–branching tree (remember $|\mathcal{F}^{\rm ult}_N| = \lambda$). Moreover, for each $p_0, \ldots, p_n \in H_0$ we have

$$\big\{\rho \in \lim(\mathcal{T}_0) : \big(\exists \zeta < \lambda\big)\big(\forall \xi > \zeta\big)\big(\rho(\xi) \notin \Sigma(p_0) \cap \ldots \cap \Sigma(p_n)\big)\big\} \in \mathbf{M}^\lambda_{\lambda,\lambda}.$$

Hence (as $|H_0| < \mathrm{cov}(\mathbf{M}^\lambda_{\lambda,\lambda})$) we may pick $\rho \in \lim(\mathcal{T}_0)$ such that for every $p_0, \ldots, p_n \in H_0$, $n < \omega$, we have

$$\big|\big\{\xi < \lambda : \rho(\xi) \in \Sigma(p_0) \cap \ldots \cap \Sigma(p_n)\big\}\big| = \lambda.$$

Let $q = \{\rho(\xi) : \xi < \lambda\}$. Then $q \in \mathbb{Q}^0_\lambda(\mathcal{F}^{\rm ult}_N) \subseteq \mathbb{Q}^0_\lambda$, $H_0 \cup \{q\}$ is linked and $q \subseteq \mathcal{D}$.

Case B: Not Case A.
Then for some $p^*_0, \ldots, p^*_m \in H_0$ and $\beta < \lambda$ we have

$$\big(\forall (\alpha, Z, d) \in \Sigma(p^*_0) \cap \ldots \cap \Sigma(p^*_m) \cap \mathcal{F}^{\rm ult}_N\big)\big(\alpha > \beta \ \Rightarrow\ (\alpha, Z, d) \notin \mathcal{D}\big).$$

It follows from the choice of N that
if $p_0, \ldots, p_n \in \mathbb{Q}^*_\lambda(\mathcal{F}^{\rm ult}_N)$ and $(\alpha, Z, d) \in \Sigma(p_0) \cap \ldots \cap \Sigma(p_n)$,
then there are Z', d' such that $(\alpha, Z', d') \in \Sigma(p_0) \cap \ldots \cap \Sigma(p_n) \cap \mathcal{F}^{\rm ult}_N$.

Consequently, we may repeat arguments of the previous case replacing in clause (ii) $\mathcal{D} \cap \mathcal{F}^{\rm ult}_N$ by $\mathcal{F}^{\rm ult}_N \setminus \mathcal{D}$. Then we obtain $q \in \mathbb{Q}^0_\lambda(\mathcal{F}^{\rm ult}_N) \subseteq \mathbb{Q}^0_\lambda$ such that $H_0 \cup \{q\}$ is linked and $q \cap \mathcal{D} = \emptyset$. □

Claim 2.10.2. *Assume that $H_0 \subseteq \mathbb{Q}^*_\lambda(\mathcal{F}^{\rm ult}_N) \subseteq \mathbb{Q}^*_\lambda$ is linked, $|H_0| < \mathrm{cov}(\mathbf{M}^\lambda_{\lambda,\lambda})$ and a sequence $\langle \delta_\xi : \xi < \lambda\rangle \subseteq \lambda$ is increasing continuously. Then there are $p \in \mathbb{Q}^0_\lambda(\mathcal{F}^{\rm ult}_N)$ and a club C^* of λ such that*

(a) *$H_0 \cup \{p\}$ is linked, and*
(b) *$\bigcup\big\{[\delta_{\xi+1}, \delta_\zeta) : \xi < \zeta$ are successive members of $C^*\big\} \in \mathrm{fil}(p)$.*

Proof of the Claim. This is essentially [8, Claim 2.14.4]. □

Now we employ a bookkeeping device to construct inductively a sequence $\langle q_\xi : \xi < 2^\lambda\rangle \subseteq \mathbb{Q}^0_\lambda(\mathcal{F}^{\rm ult}_N)$ such that

- for each $\zeta < 2^\lambda$ the family $\{q_\xi : \xi < \zeta\}$ is linked,
- if $\mathcal{D} \subseteq \mathcal{F}^{\rm ult}_N$, then for some $\xi < 2^\lambda$ we have $q_\xi \subseteq \mathcal{D}$ or $q_\xi \cap \mathcal{D} = \emptyset$,
- if $\langle \delta_\xi : \xi < \lambda\rangle \subseteq \lambda$ is increasing continuous, then for some $\varepsilon < 2^\lambda$ and a club C^* of λ we have that

$$\bigcup\big\{[\delta_{\xi+1}, \delta_\zeta) : \xi < \zeta \text{ are successive members of } C^*\big\} \in \mathrm{fil}(q_\varepsilon).$$

Since $|\mathcal{F}^{\rm ult}_N| = \lambda$, so there are no problems with carrying out the construction. It should be clear that at the end the family $\{q_\xi : \xi < 2^\lambda\}$ is linked, big and it generates a weakly reasonable ultrafilter. □

Note that we may modify the construction in the proof of Theorem 2.10 so that the resulting H is directed. Namely, by an argument similar to the one in Claim 2.10.1 we may show, that if $H_0 \subseteq \mathbb{Q}^*_\lambda(\mathcal{F}^{\rm ult}_N)$ is linked, $|H_0| < \mathrm{cov}(\mathbf{M}^\lambda_{\lambda,\lambda})$ and $p_0, p_1 \in H_0$, then there is $q \in \mathbb{Q}^0_\lambda(\mathcal{F}^{\rm ult}_N)$ such that $q \subseteq \Sigma(p_0) \cap \Sigma(p_1)$ and $H_0 \cup \{q\}$ is linked. With this claim in hands we may modify the inductive choice of $\langle q_\xi : \xi < 2^\lambda\rangle$ so that at the end $\{q_\xi : \xi < 2^\lambda\}$ is directed. However, we do not know how to guarantee the opposite, that the family $\{q_\xi : \xi < 2^\lambda\}$ is not directed

or even better, that for **no** directed $H \subseteq \mathbb{Q}^0_\lambda$ do we have $\mathrm{fil}(H) = \mathrm{fil}(\{q_\xi : \xi < 2^\lambda\})$. Thus the following question remains open.

Problem 2.11. Does "$H \subseteq \mathbb{Q}^0_\lambda$ is linked and big" imply that "H is directed"?

3. Systems of local pararegular filters

In this section we are interested in filters associated with the full system $\mathcal{F}^{\rm pr}$ of local pararegular filters on λ and we show their relation to numbers of generators (in standard sense) of some filters on λ.

Definition 3.1. Suppose that $Z \subseteq \lambda$ is an infinite set, $\alpha = \min(Z)$. A *pararegular filter on* Z is a filter F on Z such that for some system $\langle A_u : u \in [\kappa]^{<\omega}\rangle$ of sets from F we have:

- $|\omega + \alpha| \leq \kappa < \lambda$, and if $u \subseteq v \in [\kappa]^{<\omega}$, then $A_v \subseteq A_u$,
- if $U \subseteq \kappa$ is infinite, then $\bigcap\{A_{\{\xi\}} : \xi \in U\} = \emptyset$, and
- $\mathcal{F} = \big\{B \subseteq Z : \big(\exists u \in [\kappa]^{<\omega}\big)\big(A_u \subseteq B\big)\big\}$.

If the cardinal κ above satisfies $2^{|\omega+\alpha|} \leq \kappa < \lambda$, then we say that the filter F is *strongly pararegular.*

The full system of local pararegular filters on λ will be denoted by $\mathcal{F}^{\rm pr}$ and the full system of local strongly pararegular filters on λ is denoted by $\mathcal{F}^{\rm spr}$. (The latter forms a system of local filters if and only if λ is inaccessible and then $\mathcal{F}^{\rm spr} \subseteq \mathcal{F}^{\rm pr}$.)

Let us recall the following strong λ^+–chain condition.

Definition 3.2 (See Shelah [5, Def. 1.1] and [7, Def. 7])**.** Let $\mathbb{Q}$ be a forcing notion, and $\varepsilon < \lambda$ be a limit ordinal.

(1) We define a game $\Game^{\rm cc}_{\varepsilon,\lambda}(\mathbb{Q})$ of two players, Player I and Player II. A play lasts ε steps, and at each stage $\alpha < \varepsilon$ of the play sequences $\bar{p}^\alpha, \bar{q}^\alpha$ and a function φ^α are chosen so that:
- $\bar{p}^0 = \langle \emptyset_{\mathbb{Q}} : i < \lambda^+\rangle$, $\varphi^0 : \lambda^+ \longrightarrow \lambda^+ : i \mapsto 0$;
- If $\alpha > 0$, then Player I picks $\bar{p}^\alpha, \varphi^\alpha$ such that
  (i) $\bar{p}^\alpha = \langle p^\alpha_i : i < \lambda^+\rangle \subseteq \mathbb{Q}$ satisfies $(\forall \beta < \alpha)(\forall i < \lambda^+)(q^\beta_i \leq p^\alpha_i)$,
  (ii) $\varphi^\alpha : \lambda^+ \longrightarrow \lambda^+$ is regressive, i.e., $(\forall i < \lambda^+)(\varphi^\alpha(i) < 1 + i)$;
- Player II answers choosing a sequence $\bar{q}^\alpha = \langle q^\alpha_i : i < \lambda^+\rangle \subseteq \mathbb{Q}$ such that $(\forall i < \lambda^+)(p^\alpha_i \leq q^\alpha_i)$.

If at some stage of the game Player I does not have any legal move, then he loses. If the game lasted ε steps, Player I wins a play $\langle \bar{p}^\alpha, \bar{q}^\alpha, \varphi^\alpha : \alpha < \varepsilon\rangle$ if there is a club C of λ^+ such that for each distinct members i, j of C satisfying $\mathrm{cf}(i) = \mathrm{cf}(j) = \lambda$ and $(\forall \alpha < \varepsilon)(\varphi^\alpha(i) = \varphi^\alpha(j))$, the set $\{q^\alpha_i : \alpha < \varepsilon\} \cup \{q^\alpha_j : \alpha < \varepsilon\}$ has an upper bound in $\mathbb{Q}$.

(2) The forcing notion $\mathbb{Q}$ satisfies condition $(*)^\varepsilon_\lambda$ if Player I has a winning strategy in the game $\Game^{\rm cc}_{\varepsilon,\lambda}(\mathbb{Q})$.

Proposition 3.3 (See Shelah [5, Iteration Lemma 1.3] and [7, Thm 35])**.** *Let $\varepsilon < \lambda$ be a limit ordinal, $\lambda = \lambda^{<\lambda}$. Suppose that $\bar{\mathbb{Q}} = \langle \mathbb{P}_\xi, \underset{\sim}{\mathbb{Q}}_\xi : \xi < \gamma\rangle$ is a $(<\lambda)$–support iteration such that for each $\xi < \gamma$*

$$\Vdash_{\mathbb{P}_\xi} \text{“ } \underset{\sim}{\mathbb{Q}}_\xi \text{ satisfies } (*)^\varepsilon_\lambda \text{ ”}.$$

Then $\mathbb{P}_\gamma$ satisfies $(*)^\varepsilon_\lambda$.

Definition 3.4. Suppose that D is a uniform filter on λ. We define a forcing notion $\mathbb{Q}^{\rm pr}_D$ by:
a condition is a tuple $p=\big(\zeta^p,\langle\alpha^p_\xi:\xi\leq\zeta^p\rangle,\langle Z^p_\xi,F^p_\xi:\xi<\zeta^p\rangle,\mathcal{A}^p\big)$ such that

(α) $\mathcal{A}^p\subseteq D$, $|\mathcal{A}^p|<\lambda$, $\zeta^p<\lambda$,
(β) $\langle\alpha^p_\xi:\xi\leq\zeta^p\rangle$ is an increasing continuous sequence of ordinals below λ,
(γ) $Z^p_\xi=[\alpha^p_\xi,\alpha^p_{\xi+1})$ and F^p_ξ is a pararegular filter on Z^p_ξ;

the order $\leq_{\mathbb{Q}^{\rm pr}_D}=\leq$ is given by: $p\leq q$ if and only if ($p,q\in\mathbb{Q}^{\rm pr}_D$ and)

(i) $\mathcal{A}^p\subseteq\mathcal{A}^q$, $\zeta^p\leq\zeta^q$,
(ii) $\alpha^q_\xi=\alpha^p_\xi$ for $\xi\leq\zeta^p$, and $Z^p_\xi=Z^q_\xi$, $F^q_\xi=F^p_\xi$ for $\xi<\zeta^p$,
(iii) if $A\in\mathcal{A}^p$ and $\zeta^p\leq\xi<\zeta^q$, then $A\cap Z^q_\xi\in F^q_\xi$.

Proposition 3.5. *Assume $\lambda^{<\lambda}=\lambda$ and let D be a uniform filter on λ. Then:*

(1) *$\mathbb{Q}^{\rm pr}_D$ is a $({<}\lambda)$–complete forcing notion of size 2^λ,*
(2) *$\mathbb{Q}^{\rm pr}_D$ satisfies the condition $(*)^\varepsilon_\lambda$ of 3.2 for each limit ordinal $\varepsilon<\lambda$,*
(3) *if $\underset{\sim}{r}$ is a $\mathbb{Q}^{\rm pr}_D$–name such that*

$$\Vdash_{\mathbb{Q}^{\rm pr}_D}\underset{\sim}{r}=\big\{(\alpha^p_\xi,Z^p_\xi,F^p_\xi):\xi<\zeta^p\ \&\ p\in\Gamma_{\mathbb{Q}^{\rm pr}_D}\big\},$$

then $\Vdash_{\mathbb{Q}^{\rm pr}_D}$ " $\underset{\sim}{r}\in\mathbb{Q}^0_\lambda(\mathcal{F}^{\rm pr})$ and $D\subseteq{\rm fil}(\underset{\sim}{r})$ ".

Proof. (1) Note that if $\alpha<\beta<\lambda$, then there are $\leq\sum\limits_{\kappa<\lambda}2^{\kappa\cdot|[\alpha,\beta)|}$ many pararegular filters on $[\alpha,\beta)$. Hence easily $|\mathbb{Q}^{\rm pr}_D|=2^\lambda$.

If $\langle p_\alpha:\alpha<\gamma\rangle\subseteq\mathbb{Q}^{\rm pr}_D$ is $\leq_{\mathbb{Q}^{\rm pr}_D}$–increasing, $\gamma<\lambda$, then letting $\mathcal{A}^q=\bigcup\limits_{\alpha<\gamma}\mathcal{A}^{p_\alpha}$,

$$\zeta^q=\sup(\zeta^{p_\alpha}:\alpha<\gamma)\ \text{ and }\ \langle\alpha^q_\xi,Z^q_\xi,F^q_\xi:\xi<\zeta^q\rangle=\bigcup_{\alpha<\gamma}\langle\alpha^q_\xi,Z^{p_\alpha}_\xi,F^{p_\alpha}_\xi:\xi<\zeta^{p_\alpha}\rangle$$

we get a condition $q=(\zeta^q,\langle\alpha^q_\xi:\xi\leq\zeta^q\rangle,\langle Z^q_\xi,F^q_\xi:\xi<\zeta^q\rangle,\mathcal{A}^q)\in\mathbb{Q}^{\rm pr}_D$ stronger than all p_α (for $\alpha<\gamma$).

(2) Let $\mathcal{X}$ consists of all sequences $\langle Z_\xi,F_\xi:\xi<\zeta\rangle$ such that $\langle Z_\xi,F_\xi:\xi<\zeta\rangle=\langle Z^p_\xi,F^p_\xi:\xi<\zeta^p\rangle$ for some $p\in\mathbb{Q}^{\rm pr}_D$. By what we said earlier, $|\mathcal{X}|=\lambda$, so we may fix an enumeration $\langle\bar\sigma_\alpha:\alpha<\lambda\rangle$ of $\mathcal{X}$. Now, let **st** be a strategy of Player I in $\eth^{\rm cc}_{\varepsilon,\lambda}(\mathbb{Q}^{\rm pr}_D)$ which, at a stage $\alpha<\varepsilon$ of the play, instructs her to choose a legal inning $\bar p^\alpha,\varphi^\alpha$ such that if $\lambda\leq i<\lambda^+$, then $\langle Z^{p^\alpha_i}_\xi,F^{p^\alpha_i}_\xi:\xi<\zeta^{p^\alpha_i}\rangle=\bar\sigma_{\varphi^\alpha(i)}$. (Note that there are legal innings for Player I by the completeness of the forcing proved in (1) above.) Plainly, if $\langle\bar p^\alpha,\bar q^\alpha,\varphi^\alpha:\alpha<\varepsilon\rangle$ is a play of $\eth^{\rm cc}_{\varepsilon,\lambda}(\mathbb{Q}^{\rm pr}_D)$ in which Player I follows **st** and $\lambda\leq i<j<\lambda^+$ are such that $\varphi^\alpha(i)=\varphi^\alpha(j)$ for all $\alpha<\varepsilon$, then the family $\{p^\alpha_i,p^\alpha_j:\alpha<\varepsilon\}$ has an upper bound. Thus **st** is a winning strategy for Player I.

(3) Suppose $p\in\mathbb{Q}^{\rm pr}_D$ and let $\kappa=|\mathcal{A}^p|+|\omega+\alpha^p_{\zeta^p}|$. Fix a sequence $\langle A_\beta:\beta<\kappa\rangle$ listing all members of $\mathcal{A}^p\cup\{\lambda\}$ (with possible repetitions) and let $\langle u_\gamma:\gamma<\kappa\rangle$ be an enumeration of $[\kappa]^{<\omega}$. By induction on $\gamma<\kappa$ choose an increasing sequence $\langle\xi_\gamma:\gamma<\kappa\rangle\subseteq[\alpha^p_{\zeta^p},\lambda)$ such that $\xi_\gamma\in\bigcap\limits_{\beta\in u_\gamma}A_\beta$. (Remember, D is a uniform filter and $\kappa<\lambda$.) Let $\zeta^q=\zeta^p+1$, $\alpha^q_{\zeta^q}=\sup(\xi_\gamma:\gamma<\kappa)+1$ and for $u\in[\kappa]^{<\omega}$ let $B_u=\{\xi_\gamma:u\subseteq u_\gamma\ \&\ \gamma<\lambda\}$. Then $F^q_{\zeta^p}=\{B\subseteq[\alpha^q_{\zeta^p},\alpha^q_{\zeta^q}):(\exists u\in[\kappa]^{<\omega})(B_u\subseteq B)\}$ is a pararegular filter on $[\alpha^q_{\zeta^p},\alpha^q_{\zeta^q})$ and $A\cap[\alpha^q_{\zeta^p},\alpha^q_{\zeta^q})\in F^q_{\zeta^p}$ for all $A\in\mathcal{A}^q$. So

now we may take a condition $q\in\mathbb{Q}^{\rm pr}_D$ stronger than p and such that $Z^q_{\zeta^p}=[\zeta^p,\zeta^q)$, $\mathcal{A}^p=\mathcal{A}^q$. Then $q\Vdash(\alpha^q_{\zeta^p},Z^q_{\zeta^p},F^q_{\zeta^p})\in\underset{\sim}{r}$.

So we easily conclude that indeed $\Vdash_{\mathbb{Q}^{\rm pr}_D}$ " $\underset{\sim}{r}\in\mathbb{Q}^0_\lambda(\mathcal{F}^{\rm pr})$ and $D\subseteq{\rm fil}(\underset{\sim}{r})$ " (remember the definition of the order on $\mathbb{Q}^{\rm pr}_D$, specifically 3.4(iii)). $\square$

Corollary 3.6. *Assume $\lambda^{<\lambda}=\lambda$, $2^\lambda=\lambda^+$, $2^{\lambda^+}=\lambda^{++}$. Then there is a $(<\lambda)$–complete λ^+–cc forcing notion $\mathbb{P}$ such that*

$$\Vdash_{\mathbb{P}}\ \text{" } 2^\lambda=\lambda^{++} \text{ and if } D \text{ is a uniform filter on } \lambda \text{ generated by less than } \lambda^{++} \text{ elements, then } D\subseteq{\rm fil}(r) \text{ for some } r\in\mathbb{Q}^0_\lambda(\mathcal{F}^{\rm pr}) \text{ ".}$$

Proof. Using a standard bookkeeping argument build a $<\lambda$–support iteration $\bar{\mathbb{Q}}=\langle\mathbb{P}_\xi,\underset{\sim}{\mathbb{Q}}_\xi:\xi<\lambda^{++}\rangle$ such that

- for each $\xi<\lambda^{++}$ we have that $\Vdash_{\mathbb{P}_\xi}$ " $\underset{\sim}{\mathbb{Q}}_\xi=\mathbb{Q}^{\rm pr}_{\underset{\sim}{D}}$ " for some $\mathbb{P}_\xi$–name $\underset{\sim}{D}$ for a uniform filter on λ,
- if $\langle\underset{\sim}{A}_\beta:\beta<\lambda^+\rangle$ is a sequence of $\mathbb{P}_{\lambda^{++}}$–names for subsets of λ, then for some $\xi<\lambda^{++}$ such that every $\underset{\sim}{A}_\beta$ is a $\mathbb{P}_\xi$–name we have

$$\Vdash_{\mathbb{P}_\xi}\text{" if } \langle\underset{\sim}{A}_\beta:\beta<\lambda^+\rangle \text{ generates a uniform filter } D \text{ on } \lambda, \text{ then } \underset{\sim}{\mathbb{Q}}_\xi=\mathbb{Q}^{\rm pr}_D \text{ ".}$$

Now look at the limit $\mathbb{P}_{\lambda^{++}}=\lim(\bar{\mathbb{Q}})$ (and remember 3.5, 3.3). $\square$

Proposition 3.7. *Assume $2^\lambda=\lambda^+$. Then there is a uniform ultrafilter D on λ containing no ${\rm fil}(p)$ for $p\in\mathbb{Q}^*_\lambda(\mathcal{F}^{\rm pr})$.*

Proof. First note that if F is a pararegular filter on Z, then for each β we have $Z\setminus\{\beta\}\in F$. Consequently, if $\mathcal{A}\subseteq[\lambda]^\lambda$ is a family with fip, $\{[\alpha,\lambda):\alpha<\lambda\}\subseteq\mathcal{A}$, $|\mathcal{A}|\leq\lambda$, and $p\in\mathbb{Q}^*_\lambda(\mathcal{F}^{\rm pr})$, then we may choose $A\subseteq\lambda$ such that

- $\mathcal{A}\cup\{A\}$ has fip,
- for each $(\alpha,Z,F)\in p$ we have $|Z\cap A|\leq 1$ so also $Z\setminus A\in F$.

Hence, by induction on $\xi<\lambda^+$, we may choose a sequence $\langle A_\xi:\xi<\lambda^+\rangle$ of unbounded subsets of λ such that

- for $\xi<\lambda$, $A_\xi=[\xi,\lambda)$,
- $\{A_\xi:\xi<\lambda^+\}$ has fip,
- for every $A\subseteq\lambda$ there is $\xi<\lambda^+$ such that either $A_\xi\subseteq A$ or $A_\xi\cap A=\emptyset$,
- for every $p\in\mathbb{Q}^*_\lambda(\mathcal{F}^{\rm pr})$ there is $\xi<\lambda^+$ such that $\lambda\setminus A_\xi\in{\rm fil}(p)$.

Then $D=\{A\subseteq\lambda: A_{\xi_0}\cap\ldots\cap A_{\xi_n}\subseteq A$ for some $\xi_0,\ldots,\xi_n<\lambda^+$, $n<\omega\}$ is an ultrafilter as required. $\square$

Proposition 3.8. *Assume that*

(a) *there exists a λ–Kurepa tree with 2^λ λ–branches,*
(b) *D is a uniform filter on λ,*
(c) *$p\in\mathbb{Q}^0_\lambda(\mathcal{F}^{\rm pr})$ is such that ${\rm fil}(p)\subseteq D$,*
(d) *if λ is a limit cardinal, then it is strongly inaccessible and $p\in\mathbb{Q}^0_\lambda(\mathcal{F}^{\rm spr})$.*

Then the filter D cannot be generated by less than 2^λ sets, i.e., for every family $\mathcal{X}\subseteq D$ of size less than 2^λ there is a set $A\in D$ such that $|X\setminus A|=\lambda$ for all $X\in\mathcal{X}$.

Proof. Let T be a λ–Kurepa tree with 2^λ λ–branches (so each level in T is of size $<\lambda$). For $\xi<\lambda$ let T_ξ be the $\xi^{\rm th}$ level of T. Choose an increasing continuous sequence $\langle\alpha_\xi:\xi<\lambda\rangle$ such that if $(\alpha,Z,F)\in p$ and $\alpha_\xi\leq\alpha<\alpha_{\xi+1}$, then

- $Z \subseteq \alpha_{\xi+1}$ and
- there is a system $\langle A^\alpha_u : u \in [\kappa_\alpha]^{<\omega}\rangle$ of sets from F witnessing that F is pararegular (strongly pararegular if λ is inaccessible) with κ_α satisfying $|T_\xi| \leq \kappa_\alpha$.

For each $\xi < \lambda$ and $(\alpha, Z, F) \in p$ such that $\alpha_\xi \leq \alpha < \alpha_{\xi+1}$, let us fix an injection $\pi^\alpha_\xi : T_\xi \xrightarrow{1-1} \kappa_\alpha$, and next for every λ–branch η through T let us choose a set $A_\eta \in D$ so that

- if $\xi < \lambda$, $\nu \in T_\xi \cap \eta$, $(\alpha, Z, F) \in p$, $\alpha_\xi \leq \alpha < \alpha_{\xi+1}$, then $A_\eta \cap Z = A^\alpha_{\{\pi^\alpha_\xi(\nu)\}}$.

For our conclusion, it is enough to show that if $B \in D$, then there are at most finitely many λ–branches η through T such that $|B \setminus A_\eta| < \lambda$. So suppose towards contradiction $\eta_0, \eta_1, \eta_2, \dots$ are distinct λ–branches through T, $B \in D$ and $|B \setminus A_{\eta_n}| < \lambda$ for each $n < \omega$. The set $\{(\alpha, Z, F) \in p : B \cap Z \in F^+\}$ is of cardinality λ, so we may find $\xi < \lambda$ and $\nu_n \in T_\xi$ (for $n < \omega$) such that

- $\eta_n \cap T_\xi = \{\nu_n\}$ and $\nu_n \neq \nu_m$ for distinct n, m, and
- $B \cap Z^* \in (F^*)^+$ for some $(\alpha^*, Z^*, F^*) \in p$ satisfying $\alpha_\xi \leq \alpha^* < \alpha_{\xi+1}$, and
- $B \setminus \alpha_\xi \subseteq A_{\eta_n}$ for all $n < \omega$.

Then $\emptyset \neq B \cap Z^* \subseteq \bigcap\{A^\alpha_{\{\pi^\alpha_\xi(\nu_n)\}} : n < \omega\}$, a contradiction. □

4. Forcing a very reasonable ultrafilter

Our goal here is to show that the inaccessibility of λ in the assumptions of [8, Prop. 1.6(1)] is needed. This answers the request of the referee of [8] and fulfills the promise stated in [8, Rem. 1.7]. Assuming that κ is strongly inaccessible, we will construct a CS iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \kappa\rangle$ of proper forcing notions such that

$$\Vdash_{\mathbb{P}_\kappa} \text{" there is a } (\leq \omega_1)\text{–directed family } H \subseteq \mathbb{Q}^0_{\omega_1} \text{ such that } \mathrm{fil}(H) \text{ is a weakly reasonable ultrafilter on } \omega_1 \text{ and yet Odd has a winning strategy in } \eth_{\mathrm{fil}(H)} \text{ "}.$$

Let us recall the following definition.

Definition 4.1 (Shelah [8, Def. 1.4])**.** Let D be a uniform ultrafilter on λ. We define a game $\eth_D$ between two players, Odd and Even, as follows. A play of $\eth_D$ lasts λ steps and during a play an increasing continuous sequence $\bar\alpha = \langle \alpha_i : i < \lambda\rangle \subseteq \lambda$ is constructed. The terms of $\bar\alpha$ are chosen successively by the two players so that Even chooses the α_i for even i (including limit stages i where she has no free choice) and Odd chooses α_i for odd i. Even wins the play if and only if $\bigcup\{[\alpha_{2i+1}, \alpha_{2i+2}) : i < \lambda\} \in D$.

The following result was shown in [8, Prop. 1.6]:

Proposition 4.2. *Assume D is a uniform ultrafilter on λ.*

1. *If λ is strongly inaccessible and Odd has a winning strategy in $\eth_D$, then D is not weakly reasonable.*
2. *If D is not weakly reasonable, then Odd has a winning strategy in the game $\eth_D$.*

Before we define our CS iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\alpha : \alpha < \kappa\rangle$ let us introduce two main ingredients used in the construction.

Sealing the branches: At each stage of the iteration we will first use forcing notions introduced in Shelah [6, Ch. XVII, §2].

For a tree $T \subseteq {}^{<\omega_1}\omega_1$, the set of all ω_1-branches through T will be denoted by $\lim(T)$. Thus $\lim(T) = \{\eta \in {}^{\omega_1}\omega_1 : (\forall \alpha < \omega_1)(\eta{\restriction}\alpha \in T)\}$.

Lemma 4.3 (Shelah [6, Ch. XVII, Fact 2.2])**.** *Suppose that $T \subseteq {}^{<\omega_1}\omega_1$ is a tree of height ω_1. Let $\mathbb{C}$ be the Cohen forcing and $\underset{\sim}{\mathbb{L}}$ be a $\mathbb{C}$–name for the Levy collapse of $2^{\aleph_2}$ to $\aleph_1$ (with countable conditions, so it is a σ–closed forcing notion). Then $\Vdash_{\mathbb{C}*\underset{\sim}{\mathbb{L}}}$ " $\lim(T) = \big(\lim(T)\big)^{\mathbf{V}}$ ".*

Definition 4.4 (Shelah [6, Ch. XVII, Def. 2.3])**.** Suppose that $T \subseteq {}^{<\omega_1}\omega_1$ is a tree of height ω_1, $|T| = \aleph_1$, $|\lim(T)| \leq \aleph_1$. Let $\langle B_i : i < \omega_1\rangle$ list all members of $\lim(T)$ (with possible repetitions) and $\langle y_i : i < \omega_1\rangle$ list all elements of T so that $[y_j \vartriangleleft y_i \ \Rightarrow\ j < i]$. For $j < \omega_1$ we define

$$B^*_j = \begin{cases} B_i & \text{if } j = 2i, \\ \{y_i\} & \text{if } j = 2i+1, \end{cases} \qquad \text{and} \qquad B'_j = B^*_j \setminus \bigcup_{i<j} B^*_i.$$

Let $w = \{j < \omega_1 : B'_j \neq \emptyset\}$ and for $j \in w$ let $x_j = \min(B'_j)$. Finally, we put $A = \{x_i : i \in w\}$. We define a forcing notion $\mathbb{P}_T$ for *sealing the branches of T*:
a condition p in $\mathbb{P}_T$ is a finite function from $\mathrm{dom}(p) \subseteq A$ into ω such that if $\rho, \nu \in \mathrm{dom}(p)$ and $\rho \vartriangleleft \nu$, then $p(\eta) \neq p(\nu)$,
the order $\leq_{\mathbb{P}_T}$ of $\mathbb{P}_T$ is the inclusion, i.e., $p \leq q$ if and only if ($p, q \in \mathbb{P}_T$ and) $p \subseteq q$.

Lemma 4.5 (Shelah [6, Ch. XVII, Lem. 2.4])**.** *Suppose that $T \subseteq {}^{<\omega_1}\omega_1$ is a tree of height ω_1, $|T| = \aleph_1$, $|\lim(T)| \leq \aleph_1$ and $\mathbb{P}_T$ is the forcing notion for sealing the branches of T.*

(a) *$\mathbb{P}_T$ satisfies the ccc.*
(b) *If $G \subseteq \mathbb{P}_T$ is generic over $\mathbf{V}$ and $\mathbf{V}^*$ is a universe of* ZFC *extending $\mathbf{V}[G]$ and such that $(\aleph_1)^{\mathbf{V}^*} = \aleph_1^{\mathbf{V}} (= (\aleph_1)^{\mathbf{V}[G]})$, then*
$$\mathbf{V}^* \models \lim(T) = \big(\lim(T)\big)^{\mathbf{V}}.$$

Adding a bound to $\mathcal{G} \subseteq \mathbb{Q}^0_{\omega_1}$ and a family $\mathcal{U} \subseteq \mathcal{P}(\omega_1)$: After sealing branches of a tree, we will force a new member r^* of our family $H \subseteq \mathbb{Q}^0_{\omega_1}$ at the same time making sure that some family $\mathcal{U}$ of subsets of ω_1 is included in $\mathrm{fil}(r^*)$.

Definition 4.6. Suppose that $\mathcal{G} \subseteq \mathbb{Q}^0_{\omega_1}$ and $\mathcal{U} \subseteq \mathcal{P}(\omega_1)$ are such that

(a) $\mathcal{G} \subseteq \mathbb{Q}^0_{\omega_1}$ is $\leq^*$–directed and
(b) $U_0 \cap \ldots \cap U_n \in \big(\mathrm{fil}(\mathcal{G})\big)^+$ for every $U_0, \ldots U_n \in \mathcal{U}$, $n < \omega$.

We define a forcing notion $\mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})$ as follows:
a condition p in $\mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})$ is a triple $(r^p, \mathbf{G}^p, \mathbf{U}^p)$ such that $r^p \subseteq \mathcal{F}^{\mathrm{ult}}_{\omega_1}$ is countable and strongly disjoint (i.e., it satisfies the demands of 1.2(2)), $\mathbf{G}^p \subseteq \mathcal{G}$ is countable and $\mathbf{U}^p \subseteq \mathcal{U}$ is countable;
the order $\leq = \leq_{\mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})}$ is defined by: $p \leq q$ if and only if ($p, q \in \mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})$ and) $\mathbf{U}^p \subseteq \mathbf{U}^q$, $\mathbf{G}^p \subseteq \mathbf{G}^q$, $r^p \subseteq r^q$ and for every $(\alpha, Z, d) \in r^q \setminus r^p$ we have that:

- $\big(\forall (\alpha', Z', d') \in r^p\big)\big(Z' \subseteq \alpha\big)$ and
- $\big(\forall r \in \mathbf{G}^p\big)\big((\alpha, Z, d) \in \Sigma(r)\big)$ ($\Sigma(r)$ was defined in Definition 2.5) and
- $\big(\forall U \in \mathbf{U}^p\big)\big(U \cap Z \in d\big)$.

We also define a $\mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})$–name $\underset{\sim}{r}$ by $\qquad \Vdash_{\mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})}$ " $\underset{\sim}{r} = \bigcup \big\{r^p : p \in \Gamma_{\mathbb{Q}^{\mathrm{bd}}(\mathcal{G},\mathcal{U})}\big\}$ ".

Lemma 4.7. *Assume $\mathcal{G} \subseteq \mathbb{Q}^0_{\omega_1}$, $\mathcal{U} \subseteq \mathcal{P}(\omega_1)$ satisfy demands (a),(b) of 4.6. Then*

(1) $\mathbb{Q}^{\rm bd}(\mathcal{G},\mathcal{U})$ *is a σ–closed forcing notion,*
(2) $\Vdash_{\mathbb{Q}^{\rm bd}(\mathcal{G},\mathcal{U})}$ " $\underset{\sim}{r} \in \mathbb{Q}^0_{\omega_1}$ *and* $(\forall r\in\mathcal{G})(r\leq^* \underset{\sim}{r})$ *and* $\mathcal{U}\subseteq {\rm fil}(\underset{\sim}{r})$ ".

Proof. (1) Straightforward.

(2) To argue that $\Vdash_{\mathbb{Q}^{\rm bd}(\mathcal{G},\mathcal{U})}$ " $\underset{\sim}{r}\in\mathbb{Q}^0_{\omega_1}$ ", suppose $p\in\mathbb{Q}^{\rm bd}(\mathcal{G},\mathcal{U})$. Let $\{r_n : n<\omega\}=\mathbf{G}^p$, $\{U_n : n<\omega\}=\mathbf{U}^p$ (we allow repetitions). Choose inductively $(\alpha_m,Z_m,d_m)\in\mathcal{F}^{\rm ult}_{\omega_1}$ such that for $m<\omega$ we have

- $\big(\forall(\alpha',Z',d')\in r^p\big)\big(Z'\subseteq\alpha_0\big)$, $Z_m\subseteq\alpha_{m+1}$, and
- $(\alpha_m,Z_m,d_m)\in\Sigma(r_0)\cap\ldots\cap\Sigma(r_m)$, and
- $U_0\cap\ldots\cap U_m\cap Z_m\in d_m$.

[Why is the choice possible? Since $\mathcal{G}$ is directed, we may first choose $s\in\mathcal{G}$ such that $r_0,\ldots,r_m\leq^* s$. Then for some $\beta<\omega_1$, if $(\alpha,Z,d)\in s$ and $\beta\leq\alpha$, then $(\alpha,Z,d)\in\Sigma(r_0)\cap\ldots\cap\Sigma(r_m)$. By the assumption 4.6(b) on $\mathcal{U}$ we know that $U_0\cap\ldots\cap U_m\cap Z\in d$ for ω_1 many $(\alpha,Z,d)\in s$, so we may choose $(\alpha_m,Z_m,d_m)\in s$ as required.]

After the above construction is carried out, pick any uniform ultrafilter e on ω and put

$$\alpha=\alpha_0,\quad Z=\bigcup_{m<\omega} Z_m,\quad\text{and}\quad d=\bigoplus^{e}_{m<\omega} d_m.$$

Then $q=(r^p\cup\{(\alpha,Z,d)\},\mathbf{G}^p,\mathbf{U}^p)\in\mathbb{Q}^{\rm bd}(\mathcal{G},\mathcal{U})$ is a condition stronger than p. Thus by an easy density argument we see that $\Vdash_{\mathbb{Q}^{\rm bd}(\mathcal{G},\mathcal{U})}$ " $|\underset{\sim}{r}|=\omega_1$ ". The rest should be clear. □

Let us recall that a very reasonable ultrafilter on λ is a weakly reasonable ultrafilter D such that $D={\rm fil}(H)$ for some $(<\lambda^+)$–directed family $H\subseteq\mathbb{Q}^0_\lambda$ (see [8, Def 2.5(5)]). Now we may state and prove our result.

Theorem 4.8. *Assume that κ is a strongly inaccessible cardinal. Then there is a κ–cc proper forcing notion $\mathbb{P}$ such that*

$\Vdash_{\mathbb{P}}$ " *there is a $\leq^*$–increasing sequence $\langle r_\xi:\xi<\omega_2\rangle\subseteq\mathbb{Q}^0_{\omega_1}$ such that ${\rm fil}\big(\{r_\xi:\xi<\omega_2\}\big)$ is a very reasonable ultrafilter on ω_1 but Odd has a winning strategy in the game $\Game_{\{r_\xi:\xi<\omega_2\}}$* ".

Proof. The forcing notion $\mathbb{P}$ will be obtained as the limit of a CS iteration of proper forcing notions $\langle\mathbb{P}_\xi,\underset{\sim}{\mathbb{Q}}_\xi:\xi<\kappa\rangle$. The iteration will be built so that for each $\xi<\kappa$

$$\Vdash_{\mathbb{P}_\xi}\text{ " }\underset{\sim}{\mathbb{Q}}_\xi\text{ is a proper forcing notion of size }<\kappa\text{ ",}$$

so we will be sure that the intermediate stages $\mathbb{P}_\xi$ and the limit $\mathbb{P}_\kappa$ will be proper and each $\mathbb{P}_\xi$ (for $\xi<\kappa$) will have a dense subset of cardinality $<\kappa$. Thus $\mathbb{P}_\kappa$ will satisfy κ–cc (and κ will not be collapsed). Since in the process of iteration we will also collapse to $\aleph_1$ all uncountable cardinals below κ, we will know that

$$\Vdash_{\mathbb{P}_\kappa}\text{ " }\aleph_1=(\aleph_1)^{\mathbf{V}}\ \&\ 2^{\aleph_1}=\aleph_2=\kappa\text{ ".}$$

Thus we may set up a bookkeeping device that gives us a list $\langle\underset{\sim}{C}_\zeta,\underset{\sim}{A}_\zeta,\underset{\sim}{\rho}_\zeta:\zeta<\kappa\rangle$ such that

- $\underset{\sim}{C}_\zeta$ is a $\mathbb{P}_\zeta$–name for a club of ω_1,
- $\underset{\sim}{A}_\zeta$ is a $\mathbb{P}_\zeta$–name for a subset of ω_1,
- $\underset{\sim}{\rho}_\zeta$ is a $\mathbb{P}_\zeta$–name for a function from ω_1 to ω_1, and

- for each $\mathbb{P}_\kappa$–name $\underset{\sim}{C}$ for a club of ω_1, for some $\zeta<\kappa$ we have $\Vdash_{\mathbb{P}_\kappa} \underset{\sim}{C}=\underset{\sim}{C}_\zeta$, and similarly for names $\underset{\sim}{A}$ for subsets of ω_1 and names $\underset{\sim}{\rho}$ for elements of ${}^{\omega_1}\omega_1$.

Before continuing let us set some terminology used later. *A partial strategy* is a function σ such that

- $\mathrm{dom}(\sigma)\subseteq\{\eta\in{}^{<\omega_1}\omega_1:\mathrm{lh}(\eta)$ is an odd ordinal $\}$, and
- $\big(\forall\nu\in\mathrm{dom}(\sigma)\big)\big(\sigma(\nu)\in\omega_1\setminus(\sup(\nu)+1)\big)$.

We say that a sequence $\eta\in{}^{\leq\omega_1}\omega_1$ *is played according to a partial strategy* σ if

- the sequence η is increasing continuous, and
- for every odd ordinal $\alpha<\mathrm{lh}(\eta)$ we have $\eta{\restriction}\alpha\in\mathrm{dom}(\sigma)$ and $\eta(\alpha)=\sigma(\eta{\restriction}\alpha)$.

If $\rho,\eta\in{}^{\omega_1}\omega_1$ and η is played according to σ, then we say that $\eta=\sigma[\rho]$ if $\eta(0)=\rho(0)$ and $\eta(2\alpha+2)=\eta(2\alpha+1)+\rho(1+\alpha)+1$ for each $\alpha<\lambda$. Also, for an increasing sequence $\eta\in{}^{\omega_1}\omega_1$ let

$$U_\eta=\bigcup\big\{\big[\eta(2\alpha),\eta(2\alpha+1)\big):\alpha<\omega_1\big\}.$$

Now, we will inductively choose $\underset{\sim}{\mathbb{Q}}_\xi$ and $\underset{\sim}{T}_\xi,\underset{\sim}{\sigma}_\xi,\underset{\sim}{r}_\xi$ so that for each $\xi<\kappa$ the following demands are satisfied.

($\boxplus$)$_1$ $\underset{\sim}{r}_\xi$ is a $\mathbb{P}_{\xi+1}$–name for a member of $\mathbb{Q}^0_{\omega_1}$ and $\Vdash_{\mathbb{P}_{\xi+1}}(\forall\zeta<\xi)(\underset{\sim}{r}_\zeta\leq^*\underset{\sim}{r}_\xi)$,

($\boxplus$)$_2$ $\underset{\sim}{T}_\xi$ is a $\mathbb{P}_\xi$–name for a subtree of ${}^{<\omega_1}\omega_1$ of height ω_1 (with no maximal nodes).

($\boxplus$)$_3$ $\underset{\sim}{\sigma}_\xi$ is a $\mathbb{P}_\xi$–name for a partial strategy with domain $\{\eta\in\underset{\sim}{T}_\xi:\mathrm{lh}(\eta)$ is odd $\}$, and all nodes of the tree $\underset{\sim}{T}_\xi$ are played according to $\underset{\sim}{\sigma}_\xi$.

($\boxplus$)$_4$ $\Vdash_{\mathbb{P}_{\xi+1}}\big(\exists\eta\in\lim(\underset{\sim}{T}_{\xi+1})\big)\big(\eta=\underset{\sim}{\sigma}_{\xi+1}[\underset{\sim}{\rho}_\xi]\big)$.

($\boxplus$)$_5$ $\Vdash_{\mathbb{P}_\xi}(\forall\zeta<\xi)(\underset{\sim}{T}_\zeta\subseteq\underset{\sim}{T}_\xi\ \&\ \underset{\sim}{\sigma}_\zeta\subseteq\underset{\sim}{\sigma}_\xi)$ and

$\Vdash_{\mathbb{P}_{\xi+1}}$ " if $\nu\in{}^{<\omega_1}\omega_1$ is increasing continuous and such that $\mathrm{lh}(\nu)=\gamma+1$ for a limit γ and $(\forall\alpha<\gamma)(\nu{\restriction}\alpha\in\underset{\sim}{T}_\xi)$ but $\nu{\restriction}\gamma\notin\underset{\sim}{T}_\xi$, then $\nu\in\underset{\sim}{T}_{\xi+1}$ ".

($\boxplus$)$_6$ $\Vdash_{\mathbb{P}_\xi}\big(\forall\eta_0,\ldots,\eta_n\in\lim(\underset{\sim}{T}_\xi)\big)\big(\forall\zeta<\xi\big)\big(U_{\eta_0}\cap\ldots\cap U_{\eta_n}\in\big(\mathrm{fil}(\underset{\sim}{r}_\zeta)\big)^+\big)$ for each $n<\omega$, and
$\Vdash_{\mathbb{P}_{\xi+1}}\big(\forall\eta\in\lim(\underset{\sim}{T}_\xi)\big)\big(U_\eta\in\mathrm{fil}(\underset{\sim}{r}_\xi)\big)$.

($\boxplus$)$_7$ $\Vdash_{\mathbb{P}_{\xi+1}}$" $\underset{\sim}{A}_\xi\in\mathrm{fil}(\underset{\sim}{r}_\xi)$ or $\omega_1\setminus\underset{\sim}{A}_\xi\in\mathrm{fil}(\underset{\sim}{r}_\xi)$ " and $\Vdash_{\mathbb{P}_{\xi+1}}$" if $\langle\delta_\alpha:\alpha<\lambda\rangle$ is the increasing enumeration of $\underset{\sim}{C}_\xi$, then for some club $C^*\subseteq\omega_1$ we have $\omega_1\setminus\bigcup\big\{[\delta_\alpha,\delta_{\alpha+1}):\alpha\in C^*\big\}\in\mathrm{fil}(\underset{\sim}{r}_\xi)$ ".

($\boxplus$)$_8$ For $\xi>0$, $\underset{\sim}{\mathbb{Q}}_\xi$ is the $\mathbb{P}_\xi$–name for the composition

$$\mathbb{C}*\underset{\sim}{\mathbb{L}}*\mathbb{P}_{\underset{\sim}{T}_\xi}*\underset{\sim}{\mathbb{Q}}^{\mathrm{bd}}(\{\underset{\sim}{r}_\zeta:\zeta<\xi\},\{U_\eta:\eta\in\lim(\underset{\sim}{T}_\xi)\})$$

(see 4.3, 4.4, 4.6). Hence we know that also

($\boxplus$)$_9$ for every $\mathbb{P}_{\xi+1}$–name $\underset{\sim}{\mathbb{Q}}$ for a proper forcing notion, $\Vdash_{\mathbb{P}_{\xi+1}*\underset{\sim}{\mathbb{Q}}}$ " $\lim(\underset{\sim}{T}_\xi)=\big(\lim(\underset{\sim}{T}_\xi)\big)^{\mathbf{V}^{\mathbb{P}_\xi}}$ ".

To start, we let r_{-1} be any fixed element of $\mathbb{Q}^0_{\omega_1}$. We choose $\sigma':{}^{<\omega_1}\omega_1\longrightarrow\omega_1$ so that for every $\eta\in{}^{<\omega_1}\omega_1$ there is $(\alpha,Z,d)\in r$ such that $\sup(\eta)<\alpha$ and $Z\subseteq\sigma'(\eta)$, and we let $\underset{\sim}{T}_0=T_0=\{\sigma'[\rho_0]{\restriction}\alpha:\alpha<\omega_1\}\subseteq{}^{<\omega_1}\omega_1$. (So T_0 is a tree with

$\lim(T_0) = \{\sigma'[\rho_0]\}$.) Finally $\underset{\sim}{\sigma}_0 = \sigma_0 = \sigma'\restriction\{\nu \in T_0 : \mathrm{lh}(\nu) \text{ is odd }\}$. Now, the forcing notion $\mathbb{Q}_0$ is:

$$\mathbb{C} * \underset{\sim}{\mathbb{L}} * \mathbb{P}_{T_0} * \underset{\sim}{\mathbb{Q}}^{\mathrm{bd}}(\{r_{-1}\}, \{U_{\sigma_0[\rho_0]}\}).$$

Clearly, the families $\{r_{-1}\}$ and $\{U_{\sigma_0[\rho_0]}\}$ satisfy the demands (a),(b) of Definition 4.6.

Now suppose that we have arrived to a successor stage $\xi = \zeta + 1$ (and we have already defined $\mathbb{P}_\zeta$ and $\mathbb{P}_\zeta$–names $\underset{\sim}{T}_\zeta, \underset{\sim}{\sigma}_\zeta$, and $\mathbb{P}_{\varepsilon+1}$–names $\underset{\sim}{r}_\varepsilon$ for $\varepsilon < \zeta$ so that the demands of $(\boxplus)_1$–$(\boxplus)_6$ hold. It follows from $(\boxplus)_1 + (\boxplus)_6$ that $\underset{\sim}{\mathbb{Q}}_\zeta$ is correctly determined by clause $(\boxplus)_8$, so

$$\Vdash_{\mathbb{P}_\zeta} \underset{\sim}{\mathbb{Q}}_\zeta = \mathbb{C} * \underset{\sim}{\mathbb{L}} * \mathbb{P}_{\underset{\sim}{T}_\zeta} * \underset{\sim}{\mathbb{Q}}^{\mathrm{bd}}\big(\{\underset{\sim}{r}_\varepsilon : \varepsilon < \zeta\}, \{U_\eta : \eta \in \lim(\underset{\sim}{T}_\zeta)\}\big).$$

(Remember also that, by $(\boxplus)_9$, all ω_1–branches of $\underset{\sim}{T}_\zeta$ in extensions by proper forcing over $\mathbf{V}^{\mathbb{P}_\zeta * \underset{\sim}{\mathbb{Q}}_\zeta}$ are the same as those in $\mathbf{V}^{\mathbb{P}_\zeta}$.) Note, that the last factor of $\underset{\sim}{\mathbb{Q}}_\zeta$ adds an element $\underset{\sim}{r} \in \mathbb{Q}^0_{\omega_1}$ (see 4.7(2)) and we know that

$$\Vdash_{\mathbb{P}_\zeta * \underset{\sim}{\mathbb{Q}}_\zeta} \text{“ } \big(\forall \varepsilon < \zeta\big)\big(\underset{\sim}{r}_\varepsilon \leq^* \underset{\sim}{r}\big) \text{ and } \{U_\eta : \eta \in \lim(\underset{\sim}{T}_\zeta)\} \subseteq \mathrm{fil}(\underset{\sim}{r}) \text{ ”}.$$

In $\mathbf{V}^{\mathbb{P}_\zeta * \underset{\sim}{\mathbb{Q}}_\zeta}$, we may choose thin enough uncountable subset of $\underset{\sim}{r}$, getting $\underset{\sim}{r}' \subseteq \underset{\sim}{r}$ satisfying the demand in $(\boxplus)_7$ and such that

$$\big(\forall (\alpha, Z, d), (\alpha', Z', d') \in \underset{\sim}{r}'\big)\big(\alpha < \alpha' \ \Rightarrow\ \sup(Z) + \omega < \alpha'\big).$$

Let $\underset{\sim}{\sigma}' : {}^{<\omega_1}\omega_1 \longrightarrow \omega_1$ be such that $\underset{\sim}{\sigma}'\restriction\mathrm{dom}(\underset{\sim}{\sigma}_\zeta) = \underset{\sim}{\sigma}_\zeta$ and for $\nu \in {}^{<\omega_1}\omega_1 \setminus \mathrm{dom}(\underset{\sim}{\sigma}_\zeta)$ we have

$(\circledast)_1$ $\quad \underset{\sim}{\sigma}'(\nu) = \min\big\{\beta < \omega_1 : \big(\exists (\alpha, Z, d) \in \underset{\sim}{r}'\big)\big(\sup(\nu) < \alpha \ \&\ Z \subseteq \beta\big)\big\}.$

Let $\underset{\sim}{\eta}^* = \underset{\sim}{\sigma}'[\underset{\sim}{\rho}_\zeta]$ and let $\underset{\sim}{r}_\zeta = \big\{(\alpha, Z, d) \in \underset{\sim}{r}' : U_{\underset{\sim}{\eta}^*} \cap Z \in d\big\}$. It follows from our choices so far that $\underset{\sim}{r}_\zeta \in \mathbb{Q}^0_{\omega_1}$, and $\underset{\sim}{r}_\varepsilon \leq^* \underset{\sim}{r}_\zeta$ for $\varepsilon < \zeta$, and also

$(\circledast)_2$ $\quad$ for each $i < \omega_1$,

$$\underset{\sim}{\eta}^*\restriction(2i+1) \notin \underset{\sim}{T}_\zeta \ \Rightarrow\ \big(\exists (\alpha, Z, d) \in \underset{\sim}{r}_\zeta\big)\big(\underset{\sim}{\eta}^*(2i) < \alpha \ \&\ Z \subseteq \underset{\sim}{\eta}^*(2i+1)\big).$$

Put $\underset{\sim}{T}^*_\zeta = \underset{\sim}{T}_\zeta \cup \{\underset{\sim}{\eta}^*\restriction\alpha : \alpha < \omega_1\}$ and define $\underset{\sim}{\sigma}'' : {}^{<\omega_1}\omega_1 \longrightarrow \omega_1$ so that $\underset{\sim}{\sigma}''\restriction\underset{\sim}{T}^*_\zeta = \underset{\sim}{\sigma}'\restriction\underset{\sim}{T}^*_\zeta$ and for $\nu \in {}^{<\omega_1}\omega_1 \setminus \underset{\sim}{T}^*_\zeta$ we have

$(\circledast)_3$ $\quad \underset{\sim}{\sigma}''(\nu) = \min\big\{\beta < \omega_1 : \big(\exists (\alpha, Z, d) \in \underset{\sim}{r}_\zeta\big)\big(\sup(\nu) < \alpha \ \&\ Z \subseteq \beta\big)\big\}.$

Let

$$\begin{aligned} \underset{\sim}{S} = \{\nu \in {}^{<\omega_1}\omega_1 : \ & \nu \text{ is increasing continuous of length } \mathrm{lh}(\nu) = \gamma + 1 \\ & \text{for some limit } \gamma \text{ and } (\forall \alpha < \gamma)(\nu\restriction\alpha \in \underset{\sim}{T}_\zeta) \text{ but } \nu\restriction\gamma \notin \underset{\sim}{T}^*_\zeta\}. \end{aligned}$$

For each $\nu \in \underset{\sim}{S}$ let $\eta_\nu \in {}^{\omega_1}\omega_1$ be such that $\nu \vartriangleleft \eta_\nu$, η_ν is played according to $\underset{\sim}{\sigma}''$ and for every odd ordinal $\alpha \geq \mathrm{lh}(\nu)$ we have $\eta_\nu(\alpha+1) = \eta_\nu(\alpha) + 889$. Put $\underset{\sim}{T}_{\zeta+1} = \underset{\sim}{T}^*_\zeta \cup \{\eta_\nu\restriction\alpha : \nu \in \underset{\sim}{S} \ \&\ \alpha < \omega_1\}$. Note that (still in $\mathbf{V}^{\mathbb{P}_\zeta * \underset{\sim}{\mathbb{Q}}_\zeta}$) we have that $\lim\big(\underset{\sim}{T}_{\zeta+1}\big) = \lim\big(\underset{\sim}{T}_\zeta\big) \cup \{\eta_\nu : \nu \in \underset{\sim}{S}\} \cup \{\underset{\sim}{\eta}^*\}$.

It follows from the choice of $\underset{\sim}{r}$, $\underset{\sim}{r}_\zeta$ that $\big(\forall \eta \in \lim(\underset{\sim}{T}_\zeta)\big)\big(U_\eta \in \mathrm{fil}(\underset{\sim}{r}_\zeta)\big)$ and by the definition of $\underset{\sim}{\sigma}''$ we get that $\big(\forall \nu \in \underset{\sim}{S}\big)\big(U_{\eta_\nu} \in \mathrm{fil}(\underset{\sim}{r}_\zeta)\big)$ (remember the choice of $\underset{\sim}{r}'$). Hence, remembering the definition of $\underset{\sim}{r}_\zeta$, we conclude that $\big(\forall \eta \in \lim(T_{\zeta+1})\big)\big(U_\eta \in \mathrm{fil}(\underset{\sim}{r}_\zeta)\big)$.

Finally we put $\underset{\sim}{\sigma}_{\zeta+1} = \underset{\sim}{\sigma}''\restriction\{\nu \in T_{\zeta+1} : \mathrm{lh}(\nu) \text{ is odd }\}$. One easily verifies that the relevant demands in $(\boxplus)_1$–$(\boxplus)_7$ hold for $\underset{\sim}{T}_{\zeta+1}$, $\underset{\sim}{\sigma}_{\zeta+1}$ and $\underset{\sim}{r}_\zeta$. Let us also stress for future reference that

$(\circledast)_4$ if $\nu \in \mathrm{dom}(\underset{\sim}{\sigma}_{\zeta+1}) \setminus \underset{\sim}{T}_\zeta$ is of length $2i+1$, then there is $(\alpha, Z, d) \in \underset{\sim}{r}_\zeta$ such that $\nu(2i) < \alpha$ and $Z \subseteq \underset{\sim}{\sigma}_{\zeta+1}(\nu)$.

Suppose now that we have arrived to a limit stage $\xi < \kappa$ and we have defined $\mathbb{P}_\zeta$ names $\underset{\sim}{\mathbb{Q}}_\zeta, \underset{\sim}{T}_\zeta, \underset{\sim}{\sigma}_\zeta$, and $\underset{\sim}{r}_\zeta$ for all $\zeta < \xi$. In $\mathbf{V}^{\mathbb{P}_\xi}$ we define $\underset{\sim}{T}_\xi = \bigcup\limits_{\zeta<\xi} \underset{\sim}{T}_\zeta$ and $\underset{\sim}{\sigma}_\xi = \bigcup\limits_{\zeta<\xi} \underset{\sim}{\sigma}_\zeta$. We have to argue that the relevant demands in $(\boxplus)_2$–$(\boxplus)_6$ are satisfied, and the only problematic one is the first condition of $(\boxplus)_6$. If $\mathrm{cf}(\xi) = \omega_0$, then $\Vdash_{\mathbb{P}_\xi} \lim(\underset{\sim}{T}_\xi) = \bigcup_{\zeta<\xi} \lim(\underset{\sim}{T}_\zeta)$, so there are no problems. We will show that $(\boxplus)_6$ holds also if $\mathrm{cf}(\xi) \geq \omega_1$ and for this we will argue *a contrario*.

Suppose towards contradiction that ($\mathrm{cf}(\xi) \geq \omega_1$ and) we have $\mathbb{P}_\xi$–names $\underset{\sim}{\eta}_0, \ldots, \underset{\sim}{\eta}_n$ ($n < \omega$) and a condition $p \in \mathbb{P}_\xi$ such that

$$p \Vdash_{\mathbb{P}_\xi} \text{“ } \underset{\sim}{\eta}_0, \ldots, \underset{\sim}{\eta}_n \in \lim(\underset{\sim}{T}_\xi) \text{ and } \big(\exists \zeta < \xi\big)\big(\omega_1 \setminus (U_{\underset{\sim}{\eta}_0} \cap \ldots \cap U_{\underset{\sim}{\eta}_n}) \in \mathrm{fil}(\underset{\sim}{r}_\zeta)\big) \text{ ”}.$$

Remembering that $(\boxplus)_1 + (\boxplus)_6$ hold on earlier stages, we may pass to a stronger condition (if necessary) and assume additionally that for some $\zeta < \xi$, $\gamma < \omega_1$ and pairwise distinct $\nu_0, \ldots, \nu_n \in {}^\gamma\omega_1$ we have

$$\begin{array}{ll} p \Vdash_{\mathbb{P}_\xi} & \text{“ } \underset{\sim}{\eta}_0, \ldots, \underset{\sim}{\eta}_n \notin \bigcup\limits_{\varepsilon<\xi} \lim(\underset{\sim}{T}_\varepsilon) \text{ and } \underset{\sim}{\eta}_0{\restriction}\gamma = \nu_0, \ldots, \underset{\sim}{\eta}_n{\restriction}\gamma = \nu_n \text{ and} \\ & \quad \big(\forall (\alpha, Z, d) \in \underset{\sim}{r}_\zeta\big)\big(\gamma \leq \sup(Z) \ \Rightarrow\ U_{\underset{\sim}{\eta}_0} \cap \ldots \cap U_{\underset{\sim}{\eta}_n} \cap Z \notin d\big) \text{ ”}. \end{array}$$

The forcing notion $\mathbb{P}_\xi$ is proper, so we may choose a countable elementary submodel $N \prec \mathcal{H}(\chi)$ such that $\underset{\sim}{\eta}_0, \ldots, \underset{\sim}{\eta}_n, \nu_0, \ldots, \nu_n, \zeta, \xi, \gamma, p, \ldots \in N$ and then we may pick an $(N, \mathbb{P}_\xi)$–generic condition $q \geq p$. Let $\gamma^* = N \cap \omega_1$ and $\xi^* = \sup(N \cap \xi)$, and we may assume $q \in \mathbb{P}_{\xi^*}$. Then

$(\circledast)_5$ $q \Vdash_{\mathbb{P}_{\xi^*}}$“ $(\forall i \leq n)(\forall \varepsilon < \xi^*)(\exists \delta < \gamma^*)(\underset{\sim}{\eta}_i{\restriction}\delta \notin \underset{\sim}{T}_\varepsilon)$ ”, and hence $q \Vdash_{\mathbb{P}_{\xi^*}}$“ $\underset{\sim}{\eta}_0{\restriction}\gamma^*, \ldots, \underset{\sim}{\eta}_n{\restriction}\gamma^* \notin \underset{\sim}{T}_{\xi^*}$ ”.

Why? As for each $\varepsilon \in N \cap \xi$ we have a name $\underset{\sim}{\delta} \in N$ for an ordinal below ω_1 such that $p \Vdash \underset{\sim}{\eta}_i{\restriction}\underset{\sim}{\delta} \notin \underset{\sim}{T}_\varepsilon$, so we may use the genericity of q. By a similar argument,

$(\circledast)_6$ $q \Vdash_{\mathbb{P}_{\xi^*}}$“ $(\forall i \leq n)(\forall \delta < \gamma^*)(\exists \varepsilon < \xi^*)(\underset{\sim}{\eta}_i{\restriction}\delta \in \underset{\sim}{T}_\varepsilon)$ ”, so also $q \Vdash_{\mathbb{P}_{\xi^*}}$“ $(\forall \delta < \gamma^*)(\underset{\sim}{\eta}_0{\restriction}\delta, \ldots, \underset{\sim}{\eta}_n{\restriction}\delta \in \underset{\sim}{T}_{\xi^*})$ ”,

and

$(\circledast)_7$ $q \Vdash_{\mathbb{P}_{\xi^*}}$“ $\underset{\sim}{\eta}_0(\gamma^*) = \ldots = \underset{\sim}{\eta}_n(\gamma^*) = \gamma^*$ ”.

(Remember that $\underset{\sim}{\eta}_i$ are increasing continuous.) Now, consider a $\mathbb{P}_{\xi^*}$–name $\underset{\sim}{q}$ for the following member of $\underset{\sim}{\mathbb{Q}}_{\xi^*}$:

$$\big(\emptyset_{\mathbb{C}}, \emptyset_{\mathbb{L}}, \emptyset_{\mathbb{P}_{\underset{\sim}{T}_{\xi^*}}}, (\emptyset, \{\underset{\sim}{r}_\zeta\}, \emptyset)\big).$$

Directly from the definition of the order of the forcing $\mathbb{Q}^{\mathrm{bd}}$ and the choice of $\underset{\sim}{r}_{\xi^*}$ we see that

$$q \cup \{(\xi^*, \underset{\sim}{q})\} \Vdash_{\mathbb{P}_{\xi^*+1}} \text{“ } \underset{\sim}{r}_{\xi^*} \subseteq \Sigma(\underset{\sim}{r}_\zeta) \text{ ”}.$$

It follows from $(\circledast)_5, (\circledast)_6$ and $(\boxplus)_5$ that

$$q \cup \{(\xi^*, \underset{\sim}{q})\} \Vdash_{\mathbb{P}_{\xi^*+1}} \text{“ } \underset{\sim}{\eta}_0{\restriction}(\gamma^*+1), \ldots, \underset{\sim}{\eta}_n{\restriction}(\gamma^*+1) \in \underset{\sim}{T}_{\xi^*+1} \setminus \underset{\sim}{T}_{\xi^*} \text{ ”},$$

so let us look what are the respective values of the partial strategy $\underset{\sim}{\sigma}_{\xi^*+1}$. By $(\circledast)_4$ we know that

$$\begin{array}{ll} q \cup \{(\xi^*, \underset{\sim}{q})\} \Vdash_{\mathbb{P}_{\xi^*+1}} & \text{“ there exists } (A, Z, d) \in \underset{\sim}{r}_{\xi^*} \text{ such that for each } i \leq n \\ & \quad \underset{\sim}{\eta}_i(\gamma^*) = \gamma^* < \alpha \text{ and } Z \subseteq \underset{\sim}{\eta}_i(\gamma^*+1) \text{ ”}. \end{array}$$

Since $\gamma^* > \gamma$ we get a contradiction with the choice of p.

This completes the inductive definition of the iteration and the names $\underset{\sim}{T}_\xi$, $\underset{\sim}{\sigma}_\xi$ and $\underset{\sim}{r}_\xi$. It should be clear that

$$\Vdash_{\mathbb{P}_\kappa} \text{" the sequence } \langle \underset{\sim}{r}_\xi : \xi < \kappa\rangle \text{ is } \leq^*\text{–increasing and } \mathrm{fil}\big(\{r_\xi : \xi < \omega_2\}\big) \text{ is a very reasonable ultrafilter on } \omega_1 \text{ and } \bigcup_{\xi<\kappa} \underset{\sim}{\sigma}_\xi \text{ is a winning strategy for Odd in the game } \Game_{\{r_\xi:\xi<\kappa\}} \text{ ".}$$

□

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